Chapter 4

Macromechanics of Composite Laminates

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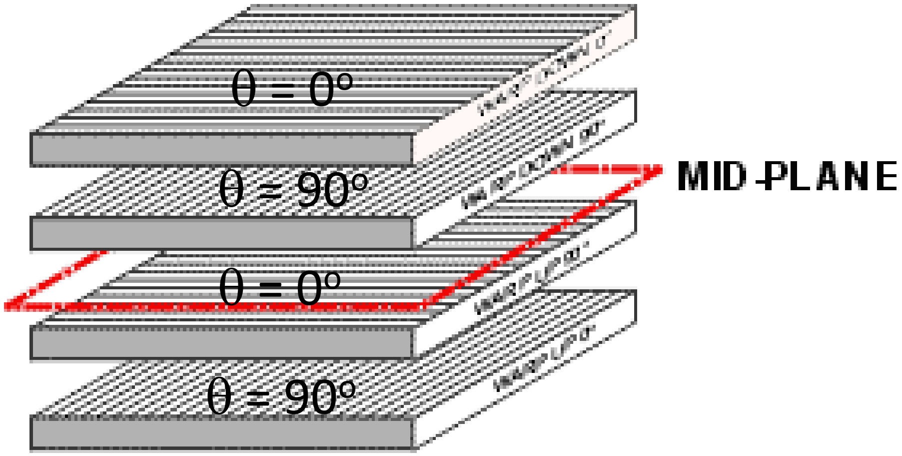
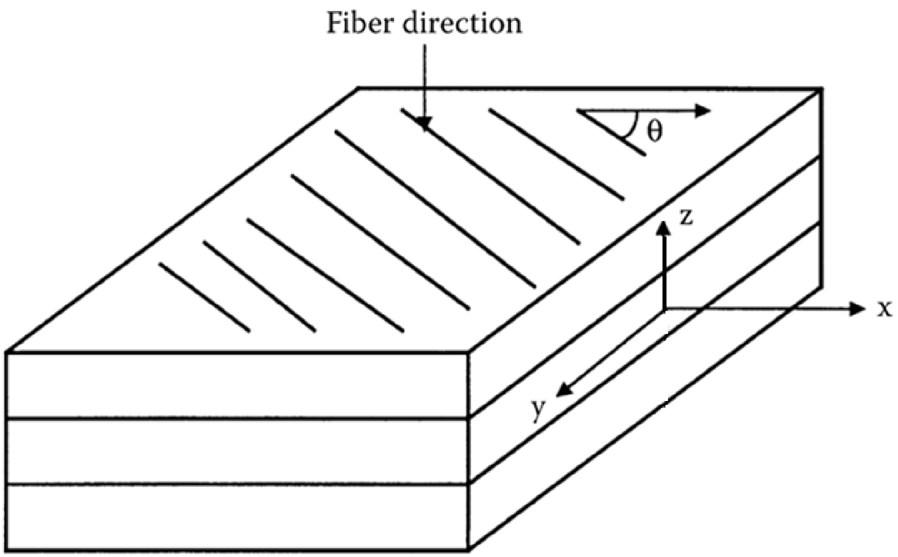
4.1. Laminate Introduction

Definition of Laminate: Two or more unidirectional laminae or plies stacked together (lay-up) at various orientations is called a **laminate**. The laminae (plies) can be of various thicknesses and consists of different or same materials.

Mid-plane: Plane forming the mid-line of the laminate.

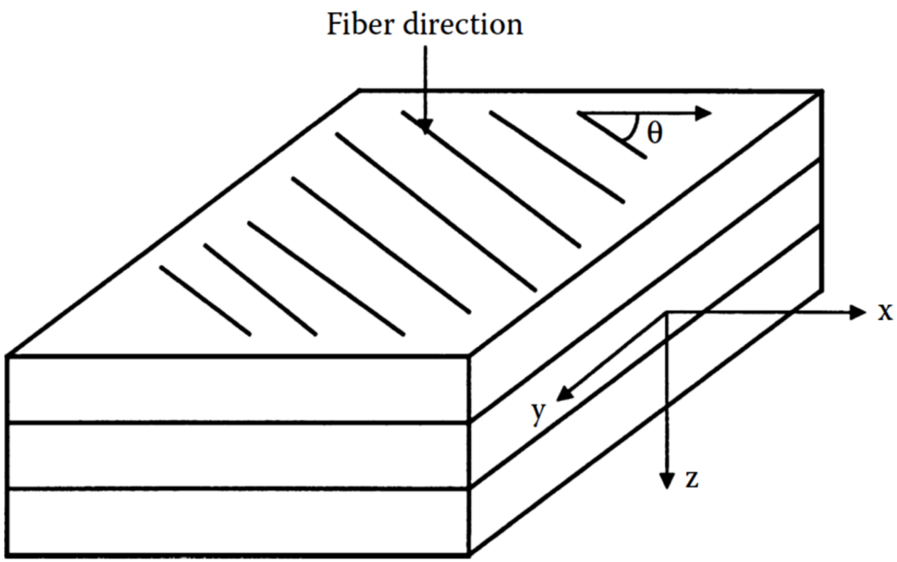


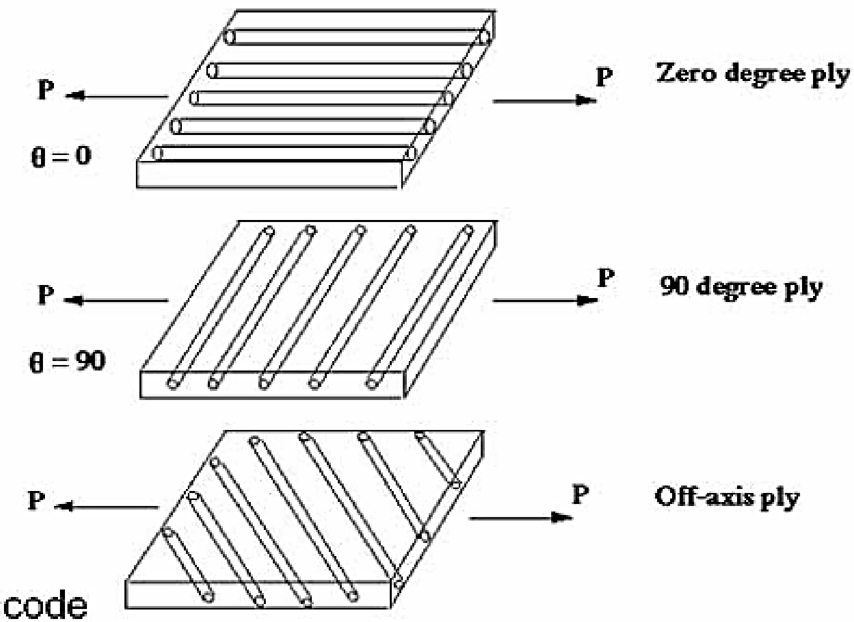


4.1. Laminate Introduction

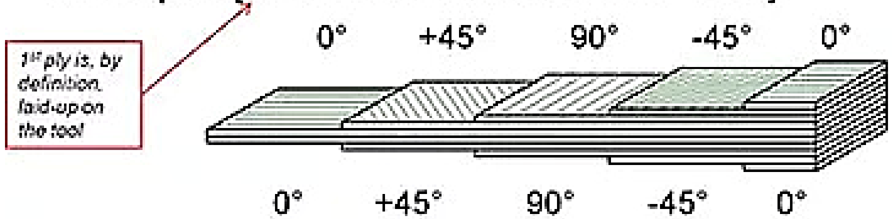
Laminate Code: layer (ply) angle

- A laminate is made of a group of single layers bonded to each other.
- Each layer can be identified by its location in the laminate, its material, and its angle of orientation with a reference axis.





- Laminates are often described by an orientation code
- Example: [0/-45/90/+45/0/0/+45/90/-45/0]



Laminate Code

0
–45
90
60
30

[0/–45/90/60/30]

or

$[0/–45/90/60/30]_T$

T stands for a total laminate.

0
–45
90
90
60
0

$[0/-45/90_2/60/0]$

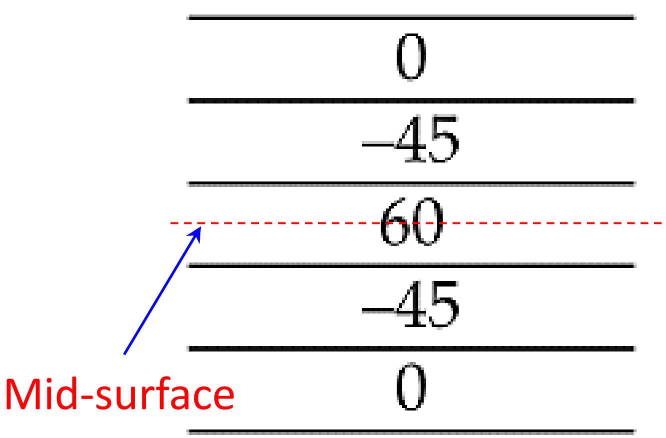
0
–45
60
60
–45
0

$[0/–45/60]_s$

Subscript s (symmetric) outside the brackets represents that the three plies are repeated in the reverse order.

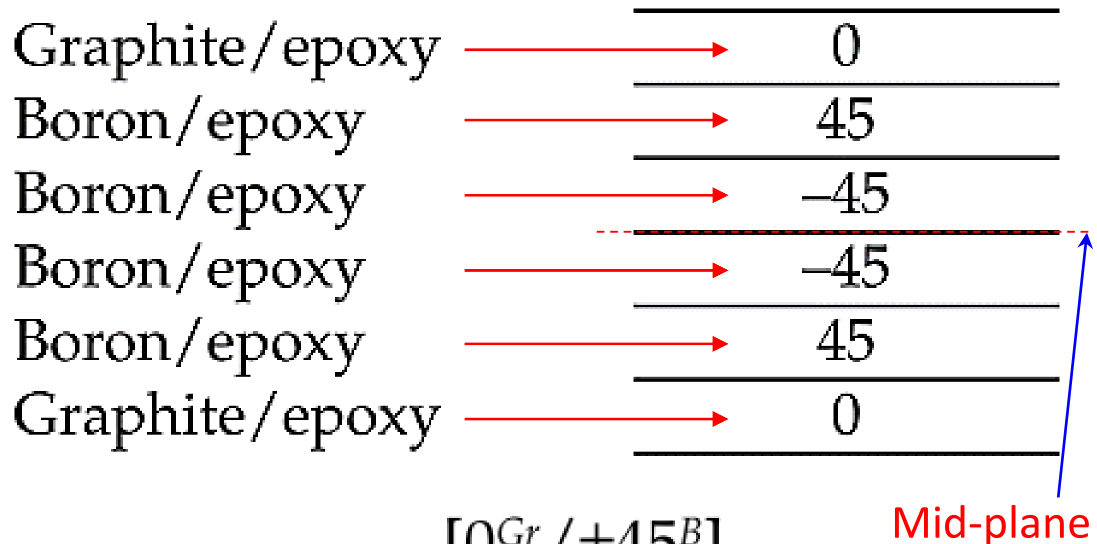
4.1. Laminate Introduction

Laminate Code



$$[0/-45/\overline{60}\,]_s$$

denotes this laminate, which consists of five plies. The number of plies is odd and symmetry exists at the mid-surface; therefore, the 60° ply is denoted with a bar on the top.



$$[0^{Gr}/\pm 45^{B}]_s$$

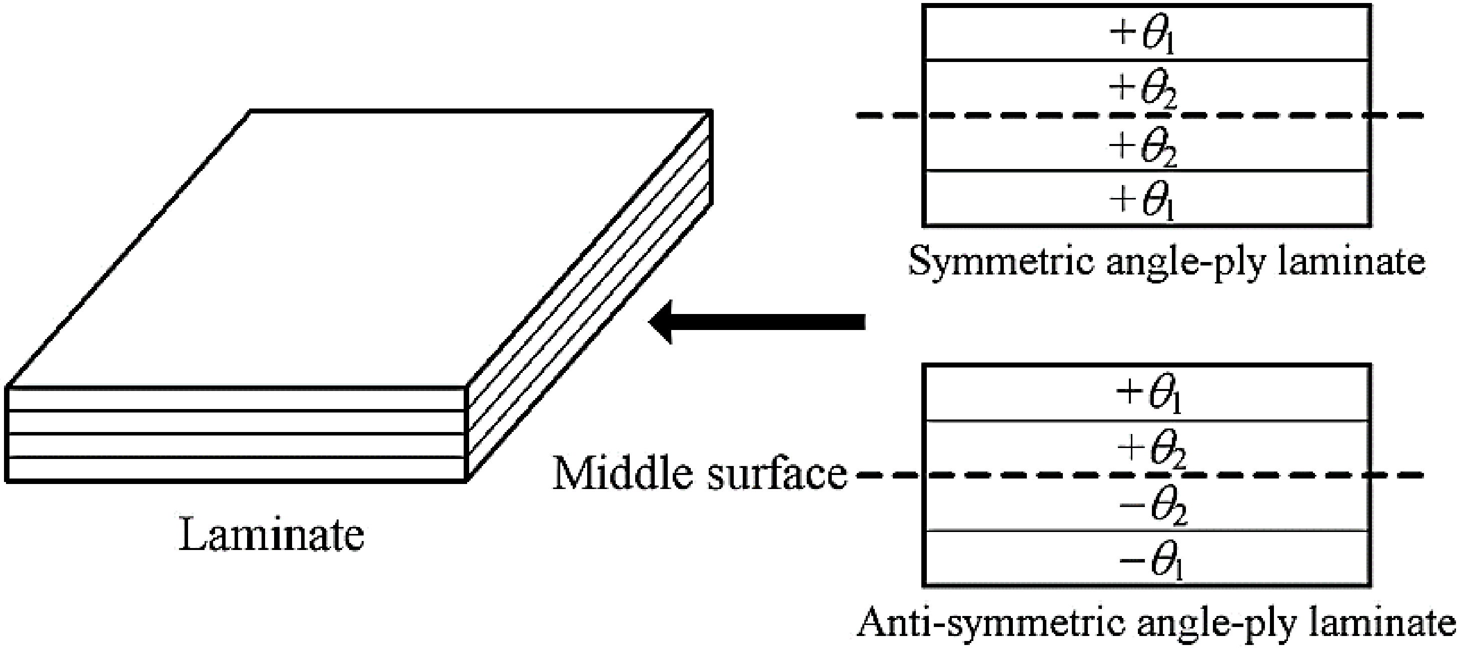
denotes the above laminate. It consists of six plies; the 0° plies are made of graphite/epoxy and the ± 45° angle plies are made of boron/epoxy. Note that s is the symmetry of the laminate.

Special Types of Laminates

- **Symmetric laminate**: for every ply above the laminate mid-plane, there is an identical ply (material and orientation) an equal distance below the mid-plane.
- **Antisymmetric laminate:** if the material and thickness of the plies are the same above and below the mid-plane, but the ply orientations at the same distance above and below the midplane are negative of each other, i.e. +45/60/-60/-45.
- **Balanced laminate:** for every ply at a +θ orientation, there is another ply at the –θ orientation somewhere in the laminate.
- **Cross-ply laminate:** composed of plies of either 0˚ or 90˚ (no other ply orientation).
- **Quasi-isotropic laminate:** are made when the orientations of the plies are balanced so that the extensional stiffness of the laminate is the same in each in-plane direction. It produced using at least three different ply orientations, all with equal angles between them. For example, quasi-isotropic laminates [0°/60°/120°] or [0°/-45°/+45°/90°].

Symmetric and anti-symmetric laminate

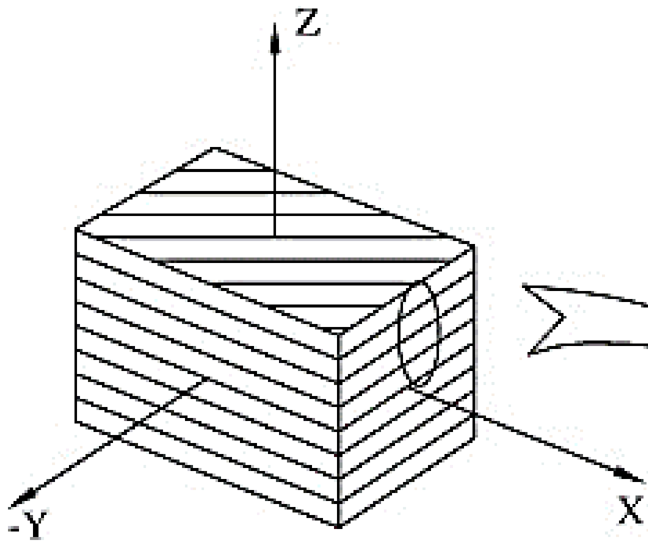
- A laminate is symmetric when the plies above the mid-plane are a mirror image of those below the mid-plane.



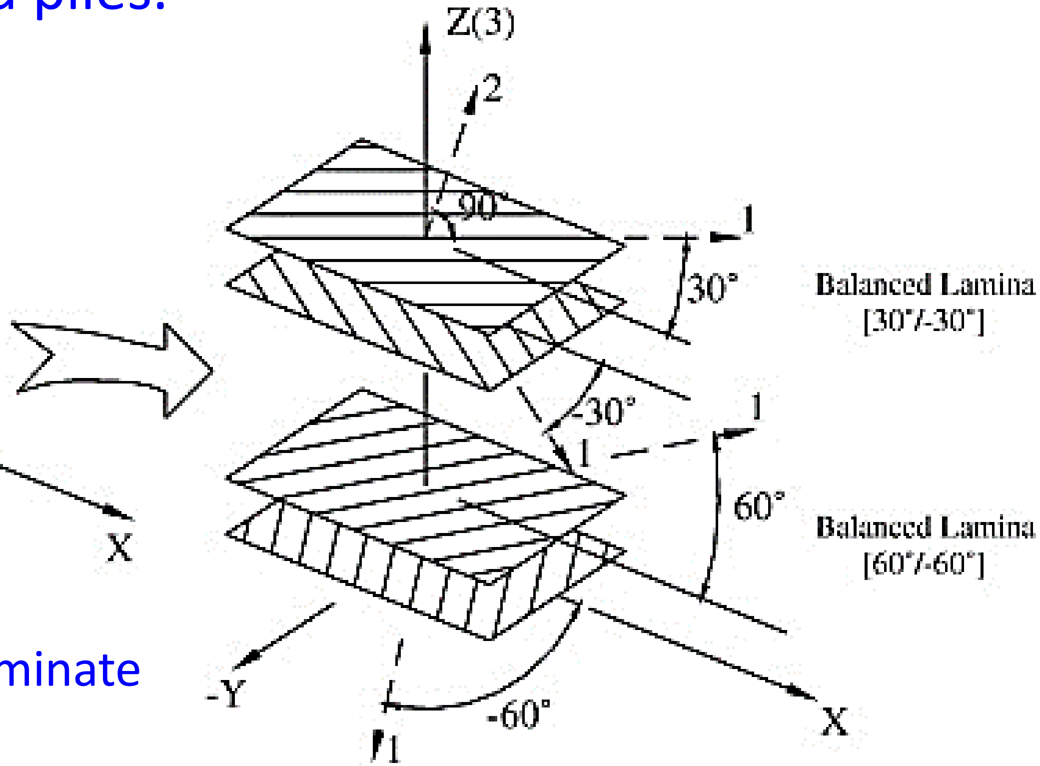
- A laminate is said to be anti-symmetric if the ply orientation angle below the mid-plane is a mirror image of the ply orientations above the mid-plane with signs reversed.

Balanced laminate

- A laminate is balanced when it has equal numbers of – (negative) and + (positive) angled plies.



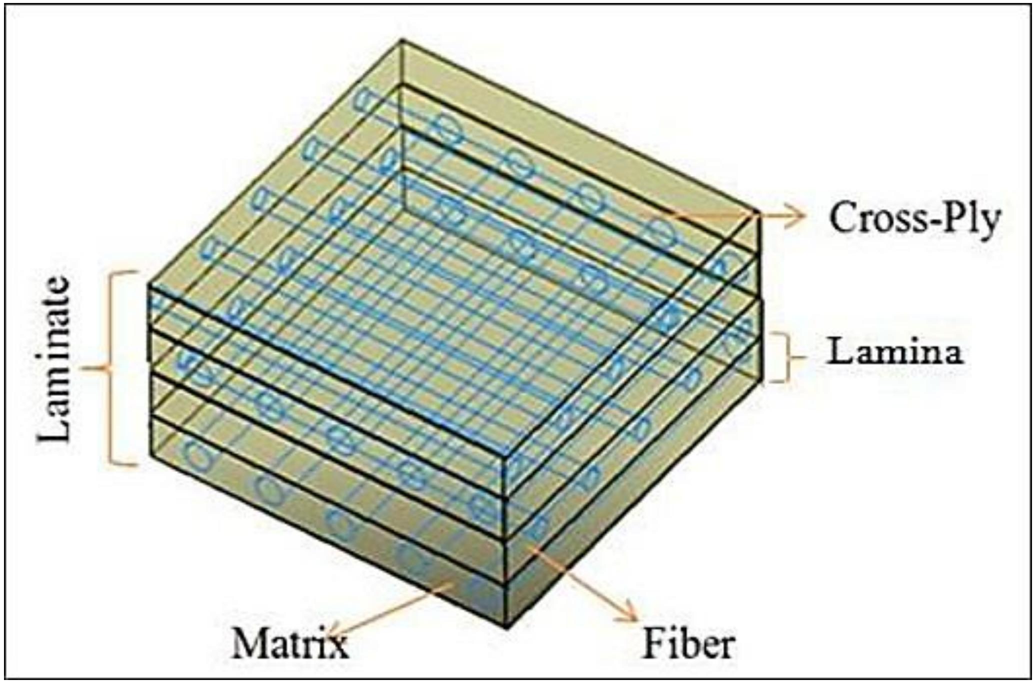
Balanced and Symmetric Laminate
[30°/-30°/60°/-60°]s



Material coordinate system: axis- 1,2,3
Structural coordinate system: axis- x,y,z

Cross-ply laminates

- A cross-ply laminate contains an arbitrary number of plies, each with a fiber orientation of either 0° or 90°, and it can be either symmetric or antisymmetric.

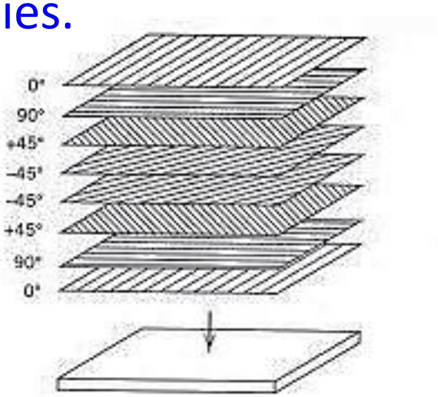


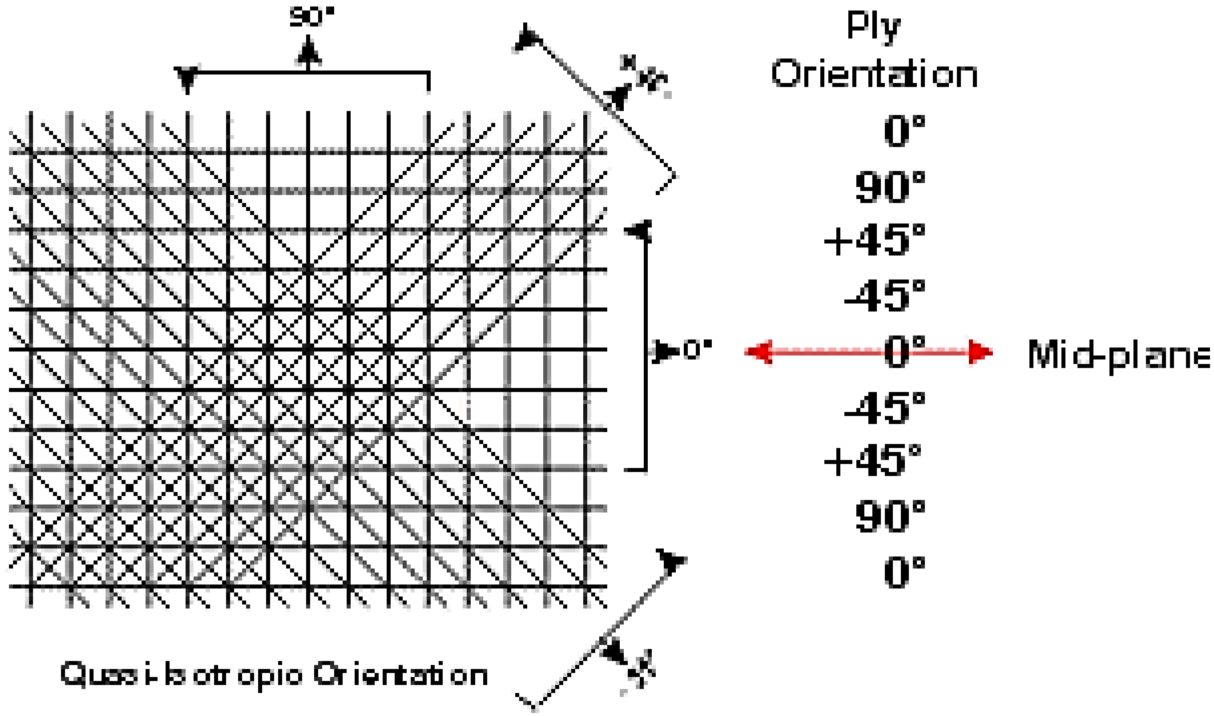
Schematic diagram of a [0/90/0/90] cross-ply laminate.

4.1. Laminate Introduction

Quasi-isotropic laminate

- Quasi-isotropic means having isotropic properties in-plane. A quasi-isotropic part has either randomly oriented fiber in all directions, or has fibers oriented such that equal strength is developed all around the plane of the part.
- A quasi-isotropic laminate made from woven fabric has plies oriented at 0º, 90º +45º and –45º. It can also be achieved with 0º, 60º and 120º oriented UD plies.





Laminates Examples

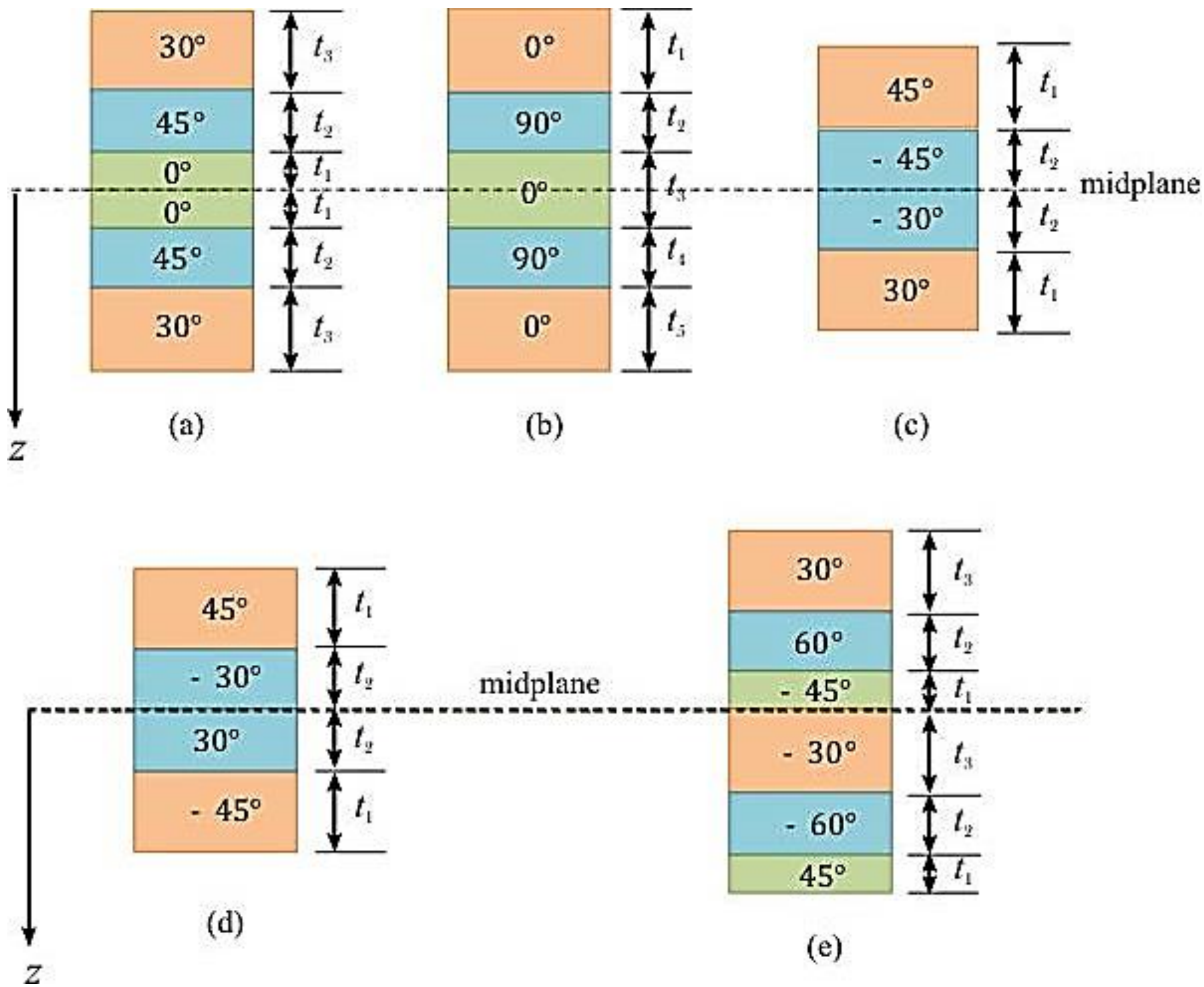
a) Symmetric laminate

b) Cross-ply laminate

c) Angle-ply laminate

d) Anti-symmetric laminate

e) Balanced laminate



Different Types Laminate Sequences

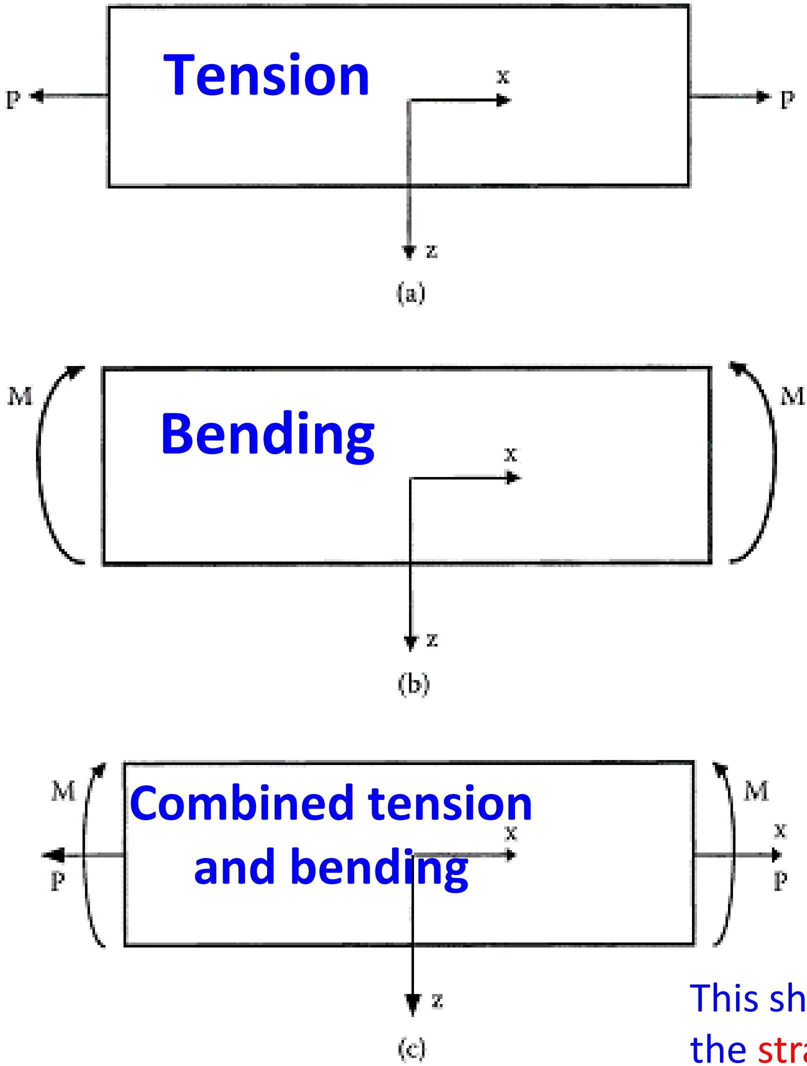
Example	Laminate Stacking Sequence
Unidirectional:	$[0/0/0/0/0/0] = [0_6]$
Cross-ply symmetric:	$[0/90/90/0] = [0/90]_s$
Angle-ply symmetric:	$[45/-45/-45/45] = [45/-45]_s = [\pm 45]_s$
Angle-ply asymmetric:	$[30/-30/30/-30/30/-30/30/-30] = [30/-30]_4 = [\pm 30]_4$
Quasi isotropic:	$[0/+45/-45/90]$
Symmetric Quasi-isotropic:	$[0/+45/-45/90/90/-45/45/0] = [0/\pm 45/90]_s$
Multidirectional:	$[0/+45/30/-30/45]$
Hybrid:	$[0^k/0^k/45^c/-45^c/90^G/-45^c/45^c/0^k/0^k]_T = [0_2^k/\pm 45^c/90^G]_s$

Define: S=Symmetric, K=Kevlar, C=Carbon, G=Glass, T=Total

Hybrid Composites: A composite made up of 2 or more different types of materials.

4.2. Stress-strain Relations for a Laminate

Stress - Strain Relations for an Isotropic Beam (Euler–Bernoulli beam)



$$\sigma_x = \frac{P}{A}, \quad \varepsilon_x = \frac{P}{AE}$$

$$\sigma_{xx} = \frac{Mz}{I}, \quad \varepsilon_{xx} = \frac{z}{\rho} = \frac{Mz}{EI}$$

$$\varepsilon_{xx} = \left(\frac{1}{AE}\right)P + \left(\frac{z}{EI}\right)M$$

$$= \varepsilon_0 + z\left(\frac{1}{\rho}\right) = \varepsilon_0 + z\kappa$$

This shows that, under a combined uniaxial and bending load, the strain varies linearly through the thickness of the beam

Classical Lamination Theory

- The classical lamination theory (CLT) is used to develop the stress-strain relations for a laminate. CLT is a direct extension of the classical plate theory for isotropic and homogeneous material as proposed by Kirchhoff –Love. The extension of this theory to laminates requires modifications to take into account the inhomogeneity in thickness direction.
- CLT is a commonly used predictive tool which makes it possible to analyze complex coupling effects that may occur in composite laminates.
- CLT is able to predict strains, displacements and curvatures that develop in a laminate as it is mechanically and thermally loaded.
- Use CLT to develop similar relationships in 3D for a laminate (plate) under combined shear and axial forces and bending and twisting moments.

Assumptions of Classical Lamination Theory

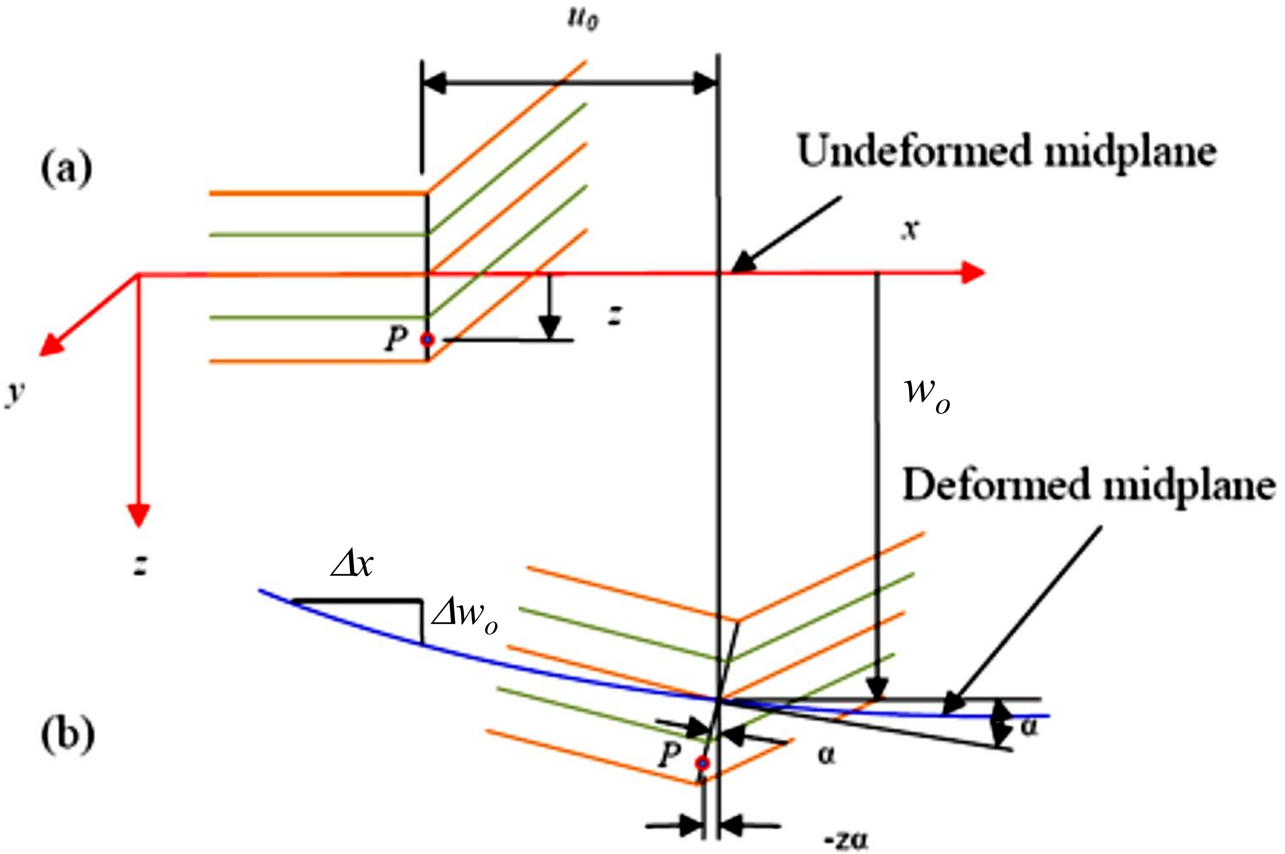
1. Each lamina is orthotropic and homogeneous.
2. The laminate deforms according to the Kirchhoff - Love assumptions for bending and stretching of thin plates (as assumed in classical plate theory).
3. The laminate is thin and is loaded only in its plane (plane stress) $\left(\sigma_z = \tau_{xz} = \tau_{yz} = 0\right)$
4. Displacements are continuous and small throughout the laminate $|u|, |v|, |w| \ll |h|,$ where h is the laminate thickness
5. Each lamina is elastic (stress-strain relations are linear).
6. The laminate consists of perfectly bonded laminae. There is no slip that occurs between the lamina interfaces.

Two approaches of 2nd assumption

1. The normals to the mid-plane remain straight and normal to the mid-plane even after deformation, results into zero transverse shear strains: $\gamma_{xz} = \gamma_{yz} = 0$

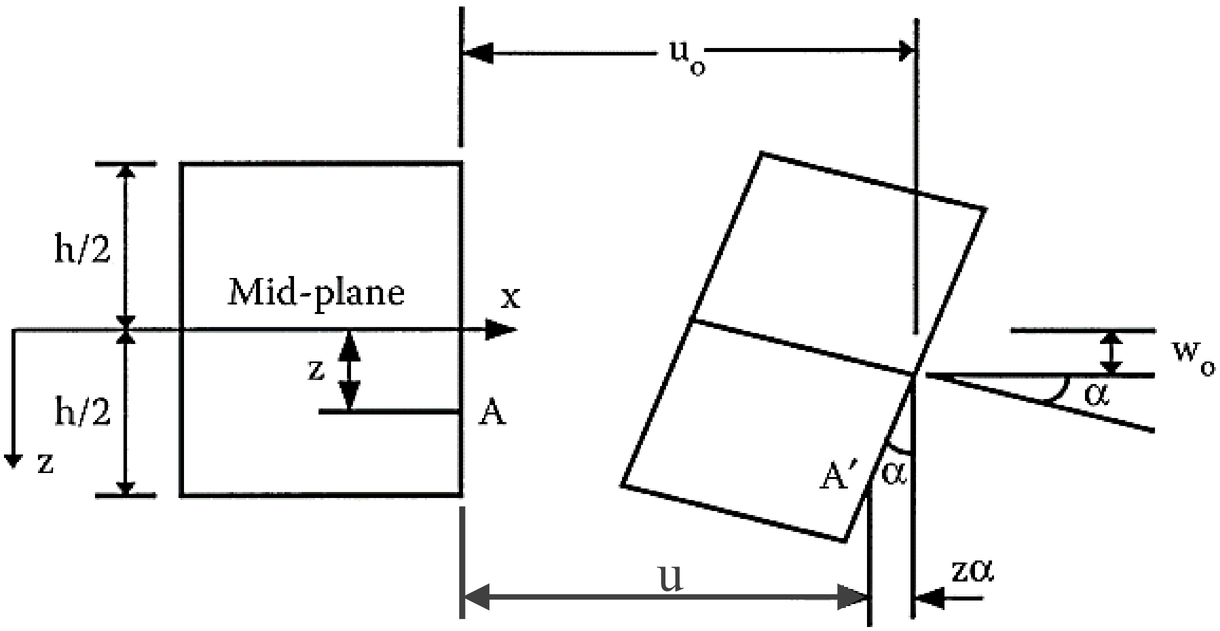
2. The normals to the mid-plane do not change their lengths, results into zero transverse normal strain.

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$



Kinematics of CLT: Displacement field (Kirchhoff–Love theory)

u_0, v_0, w_0: displacements at the mid-plane in the x, y & z directions, respectively.
u, v, w: the displacements at any point in the x, y, & z directions, respectively.



$$u(x,y,z) = u_0(x,y) - z\alpha$$

$$\alpha = \frac{\partial w_0}{\partial x}$$

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x}$$

$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y}$$

$$w(x,y,z) = w_0(x,y)$$

Kinematics of CLT: Strain-Displacement Equations

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x}$$

$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y}$$

$$w(x,y,z) = w_0(x,y)$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} \qquad \varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x \partial y}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + z\begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}.$$

Strain-Displacement Equations

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}.$$

Curvatures in the laminate

Distance from the mid-plane to the point in the thickness direction

Global strains

Mid-plane strains in the laminate

Strain and Stress of each lamina in a Laminate

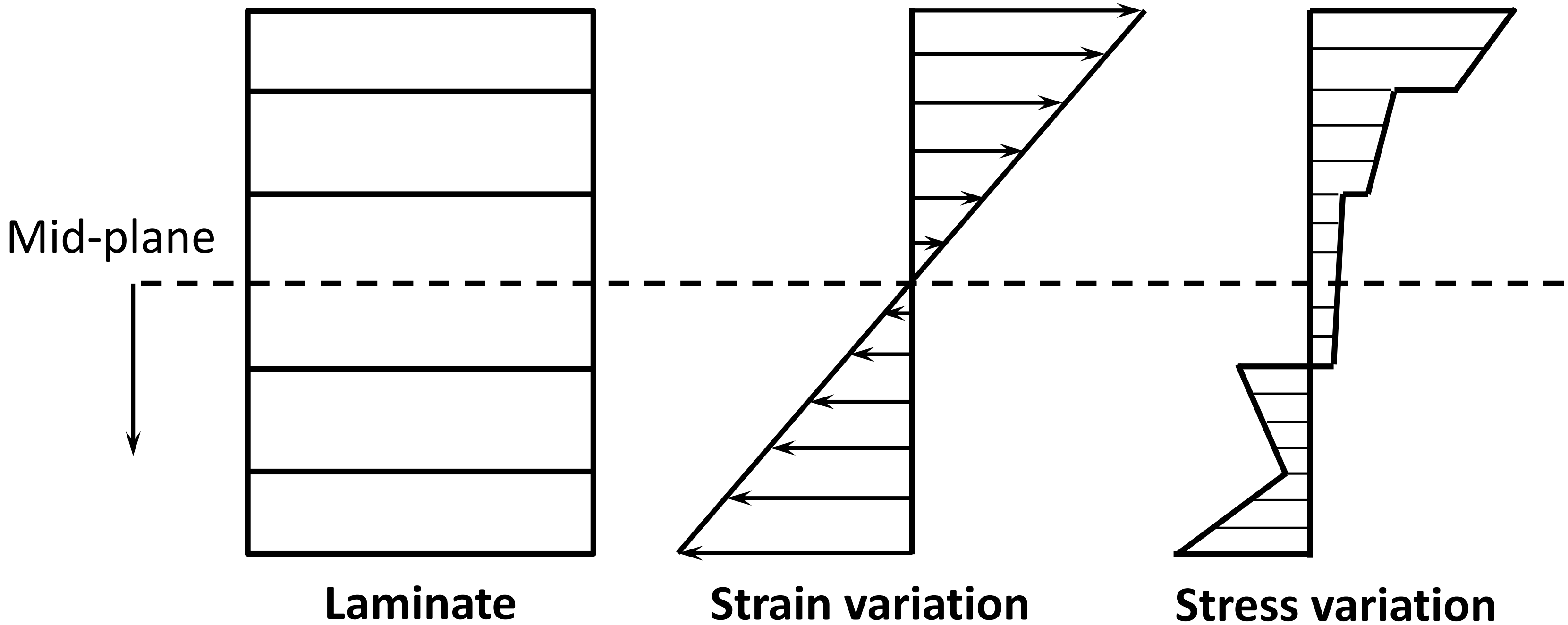
$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k$$

$$\boxed{\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}.}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}.$$

4.2. Stress-strain Relations for a Laminate

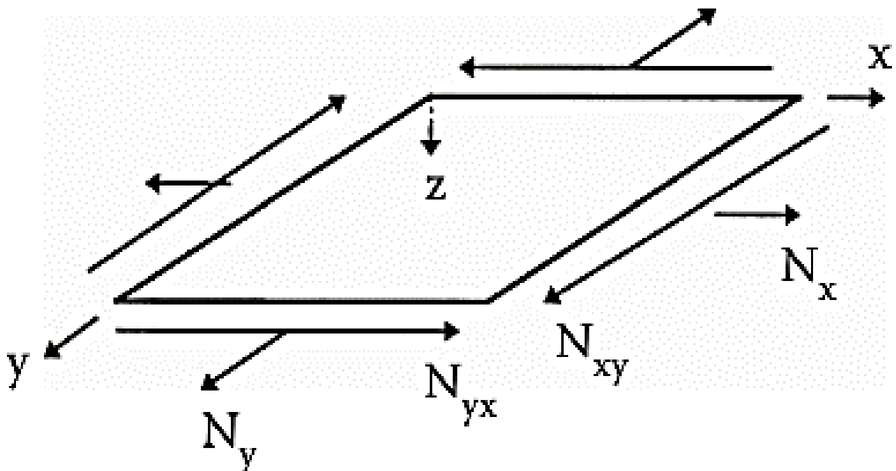
Strain and stress variation through the laminate thickness.

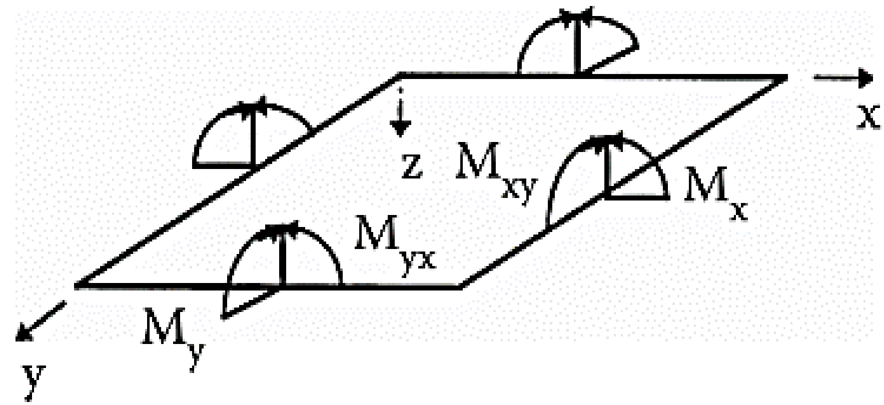


- The strains vary linearly over the thickness of a laminate.
- The stresses vary linearly only through the thickness of each lamina.
- The stresses may jump from lamina to lamina since the transformed reduced stiffness matrix changes from ply to ply.

Kinetics of CLT: Force and Moment Resultants on a Laminate

Consider the general case of a plate under in-plane shear and axial loading, as well as bending and twisting moments.





N_x = normal force resultant in the x direction (per unit length)
N_y = normal force resultant in the y direction (per unit length)
N_{xy} = shear force resultant (per unit length)

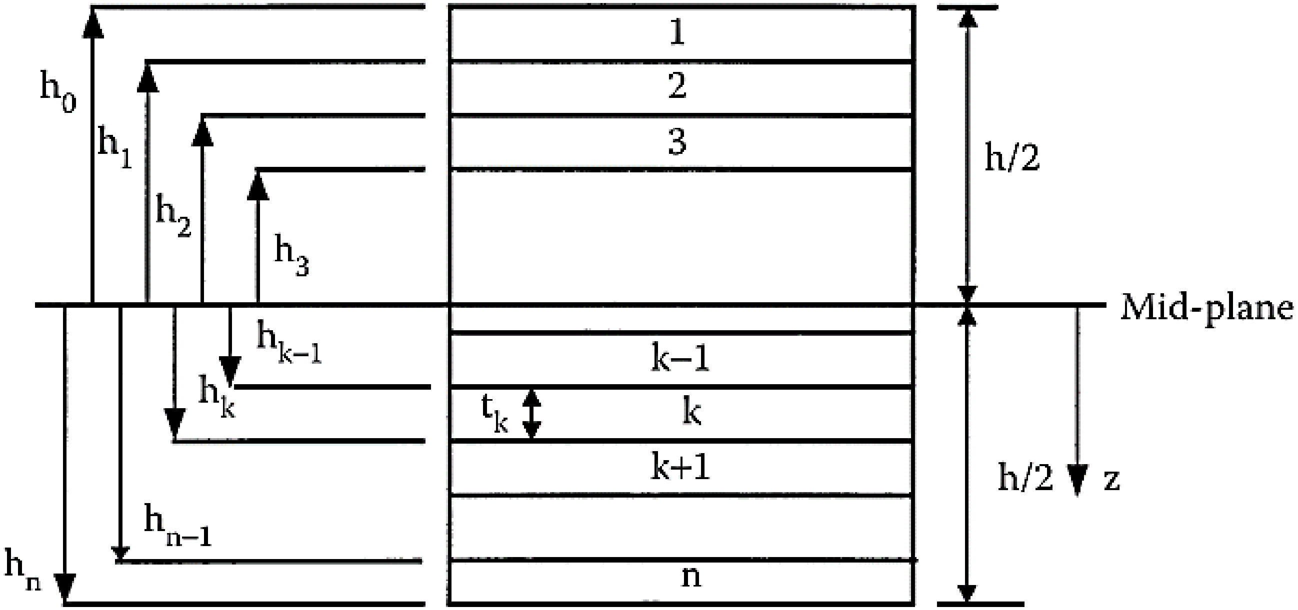
M_x = bending moment resultant in the yz plane (per unit length)
M_y = bending moment resultant in the xz plane (per unit length)
M_{xy} = twisting moment resultant (per unit length)

Force and Moment Resultants

- The stresses in each lamina can be integrated to give resultant forces and moments (or applied forces and moments.).
- Since the forces and moments applied to a laminate will be known, the mid-plane strains and plate curvatures can then be found.
- Consider a laminate made of *n* plies as shown, each ply has a thickness t_k.
- The location of the mid-plane is h/2 from the top or bottom surface.

The thickness of the laminate *h* is

$$h = \sum_{k=1}^{n} t_k .$$



4.2. Stress-strain Relations for a Laminate

Coordinate Locations of Plies in a Laminate

The z-coordinate of each ply k surface (top and bottom) is given by

Ply 1: $h_0 = -\dfrac{h}{2}\ (top\ \ surface),\qquad h_1 = -\dfrac{h}{2} + t_1\ (bottom\ \ surface)\ .$

Ply k: (k = 2, 3,…n – 2, n – 1):

$$h_{k-1} = -\frac{h}{2} + \sum_{=1}^{k-1} t\ \ (top\ \ surface)\qquad h_k = -\frac{h}{2} + \sum_{=1}^{k} t\ \ (bottom\ \ surface)\ .$$

Ply n: $h_{n-1} = \dfrac{h}{2} - t_n\ (top\ \ surface)\qquad h_n = \dfrac{h}{2}\ (bottom\ \ surface)\ .$

Force and Moment Resultants

Integrating the global stresses in each lamina gives the resultant forces per unit length in the x–y plane through the laminate thickness as

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz, \qquad N_y = \int_{-h/2}^{h/2} \sigma_y dz, \qquad N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz,$$

Similarly, integrating the global stresses in each lamina gives the resulting moments per unit length in the x–y plane through the laminate thickness as

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz, \qquad M_y = \int_{-h/2}^{h/2} \sigma_y z dz, \qquad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz,$$

Force and Moment Resultants

The resultant forces and moments can be written in matrix form:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz, \quad = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz,$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} zdz, \quad = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k zdz,$$

Force and Moment Resultants

The resultant forces and moments can be written in terms of the mid-plane strains and curvatures:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} dz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} z\, dz$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} z dz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} z^2\, dz$$

Force and Moment Resultants

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \underbrace{\left\{ \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} dz \right\}}_{[A]} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \underbrace{\left\{ \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \int_{h_{k\;1}}^{h_k} z\, dz \right\}}_{[B]} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \underbrace{\left\{ \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z dz \right\}}_{[B]} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \underbrace{\left\{ \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \int_{h_{k\;1}}^{h_k} z^2\, dz \right\}}_{[D]} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$\int_{h_{k-1}}^{h_k} dz = (h_k - h_{k-1}), \quad \int_{h_{k-1}}^{h_k} z dz = \frac{1}{2}(h_k^2 - h_{k-1}^2), \quad \int_{h_{k-1}}^{h_k} z^2 dz = \frac{1}{3}(h_k^3 - h_{k-1}^3),$$

Force and Moment Resultants

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$\text{or } [N] = [A]\left[\varepsilon^o\right] + [B][\kappa]$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$\text{or } [M] = [B]\left[\varepsilon^o\right] + [D][\kappa]$$

Force and Moment Resultants

[A] – Extensional stiffness matrix relating the resultant in-plane forces to the in-plane strains.

$$A_{ij} = \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \ (h_k - h_{k-1}), \ i = 1,2,6; \ j = 1,2,6,$$

[B] – Coupling stiffness matrix coupling the force and moment terms to the mid-plane strains and mid-plane curvatures.

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \ (h_k^2 - h_{k-1}^2), \ i = 1,2,6; \ j = 1,2,6$$

[D] – Bending stiffness matrix relating the resultant bending moments to the plate curvatures.

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \ (h_k^3 - h_{k-1}^3), \qquad i = 1,2,6; \ j = 1,2,6.$$

Analysis of Composite Laminates

Combine into one general expression for a composite laminate

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

or $\begin{bmatrix} \underline{N} \\ M \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} \\ B & D \end{bmatrix} \begin{bmatrix} \dfrac{\varepsilon^0}{\kappa} \end{bmatrix}$

$$[N] = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} \quad [M] = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} \quad [\varepsilon^0] = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \quad [\kappa] = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

Special Cases of Laminate

- Symmetric Laminate: $[B] = 0$. The force and moment equations can be decoupled.

$$[N]=[A]\left[\varepsilon^{o}\right]; \quad [M]=[D][\kappa]$$

- Balanced Laminate: $A_{16} = A_{26} = 0$
- Symmetric and Balanced Laminate: $[B] = 0$ and $A_{16} = A_{26} = 0$

Orthotropic with respect to in-plane behavior.

$$\begin{bmatrix} N_x \\ N_y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \end{bmatrix}$$

$$N_{xy} = A_{66}\gamma_{xy}^0$$

Special Cases of Laminate

- Cross-Ply Laminate: $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0$.

Some decoupling of the six equations.

$$\begin{bmatrix} N_x \\ N_y \\ M_x \\ M_y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{12} & A_{22} & B_{12} & B_{22} \\ B_{11} & B_{12} & D_{11} & D_{12} \\ B_{12} & B_{22} & D_{12} & D_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \kappa_x \\ \kappa_y \end{bmatrix}$$

$$\begin{bmatrix} N_{xy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{66} & B_{66} \\ B_{66} & D_{66} \end{bmatrix} \begin{bmatrix} \gamma_{xy}^0 \\ \kappa_{xy} \end{bmatrix}$$

Orthotropic with respect to both in-plane and bending behavior.

Special Cases of Laminate

- Symmetric and Cross-Ply Laminate: [*B*] = 0 and

$A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0.$

Significant decoupling

$$\begin{bmatrix} N_x \\ N_y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \end{bmatrix} \qquad \begin{bmatrix} M_x \\ M_y \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix}$$

$$N_{xy} = A_{66} \gamma_{xy}^0 \qquad M_{xy} = D_{66} \kappa_{xy}$$

Orthotropic with respect to both in-plane and bending behavior.

Special Cases of Laminate

- Quasi-Isotropic Laminate:

A quasi-isotropic laminate is isotropic in the xy-plane. In such a laminate, the extensional stiffness matrix behaves like that of an isotropic material, which implies

$$A_{11} = A_{22}$$
$$A_{16} = A_{26} = 0$$
$$A_{66} = (A_{11} - A_{12})/2$$

Note that the other two stiffness matrices [B] and [D] may not behave like isotropic materials. Some examples of quasi-isotropic laminates are

$$[0°/30°/60°/90°], [0°/60°/-60°], [0°/45°/90°/-45°], \text{etc.}$$

4.2. Stress-strain Relations for a Laminate

Analysis Procedures for Laminated Composites (Steps)

1. Find the value of the reduced stiffness matrix [Q] for each ply using its four elastic moduli, E_1 , E_2 , ν_{12} , and G_{12}.
2. Find the value of the transformed reduced stiffness matrix $[\bar{Q}]$ for each ply using the [Q] matrix calculated in step 1 and the angle of the ply .
3. Knowing the thickness, t_k , of each ply, find the coordinate of the top and bottom surface, h_i , i = 1..., n, of each ply.
4. Use the $[\bar{Q}]$ matrices from step 2 and the location of each ply from step 3 to find the three stiffness matrices [A], [B], and [D].

Analysis Procedures for Laminated Composites (Steps)

5. Substitute the stiffness matrix values found in step 4 and the applied forces and moments.
6. Solve the six simultaneous equations to find the mid-plane strains and curvatures.
7. Now that the location of each ply is known, find the global strains in each ply.
8. For finding the global stresses, use the stress–strain relation.
9. For finding the local strains, use the transformation.
10. For finding the local stresses, use the transformation.

Example 1

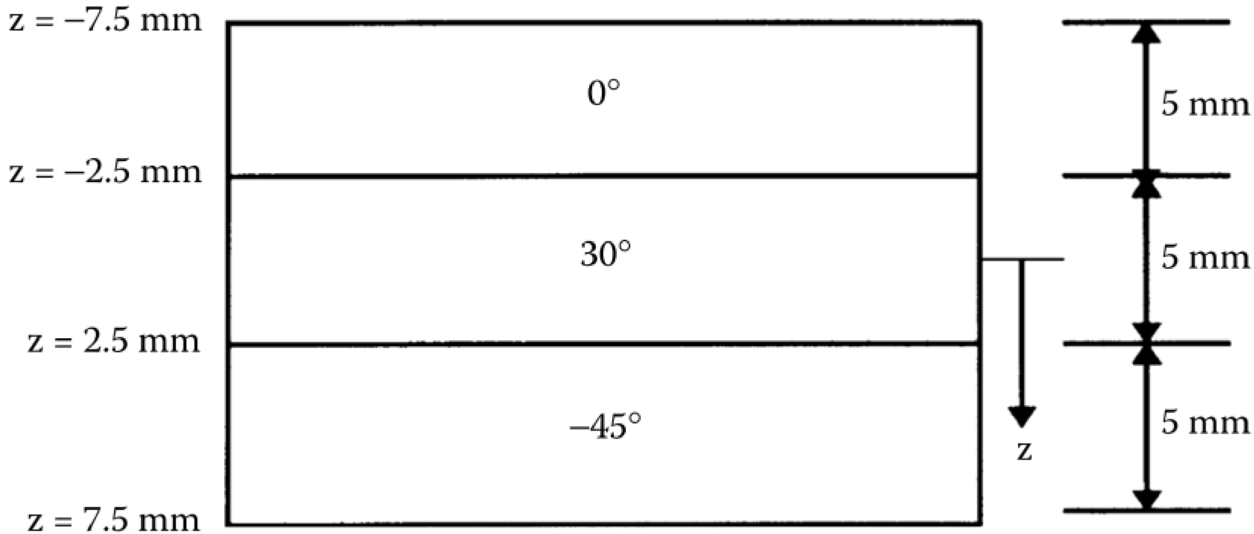
A [0/30/-45] Graphite/Epoxy laminate is subjected to a load of $N_x = N_y = 1000$ N/m. Use the unidirectional properties from Table 2.1 of Graphite/Epoxy (A.K. Kaw, Mechanics of Composite Materials, 2nd edition, Taylor & Francis, 2006). Assume each lamina has a thickness of 5 mm. Find

a) the three stiffness matrices [A], [B] and [D],

b) mid-plane strains and curvatures,

c) global and local stresses on top surface of 30^0 ply,

d) percentage of load N_x taken by each ply.



Solution

a) The three stiffness matrices [A], [B] and [D]

Step 1: Find the reduced stiffness matrix [Q] for each ply

$$Q_{11} = \frac{E_1}{1 - \nu_{21}\ \nu_{12}}$$

$$Q_{12} = \frac{\nu_{12}\ E_2}{1 - \nu_{21}\nu_{12}}$$

$$Q_{22} = \frac{E_2}{1 - \nu_{21}\ \nu_{12}}$$

$$Q_{66} = G_{12}$$

$$[Q] = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)\ \text{Pa}$$

Step 2: Find the transformed stiffness matrix $\left[\bar{Q}\right]$ using the reduced stiffness matrix [Q] and the angle of the ply.

$$\bar{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^4)$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})s^3c$$

$$\bar{Q}_{22} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4)$$

The transformed reduced stiffness matrix for each of the three plies are

$$[\bar{Q}]_0 = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)\ \mathrm{Pa}$$

$$[\bar{Q}]_{30} = \begin{bmatrix} 109.4 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix} (10^9)\ \mathrm{Pa}$$

$$[\bar{Q}]_{-45} = \begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix} (10^9)\ \mathrm{Pa}$$

4.2. Stress-strain Relations for a Laminate

Step 3: Find the coordinate of the top and bottom surface of each ply.

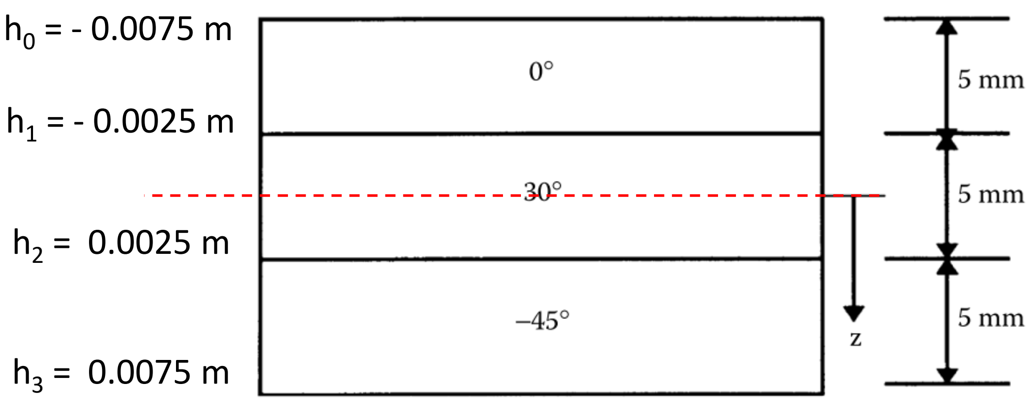
Ply n: $h_{n-1} = \frac{h}{2} - t_n \ (top \ \ surface)$ $\qquad h_n = \frac{h}{2} \ (bottom \ \ surface)$.

The total thickness of the laminate is

$$h = (0.005)(3) = 0.015 \text{ m}.$$

The mid plane is 0.0075 m from the top and bottom of the laminate. Hence using Equation (4.20), the location of the ply surfaces are

h_0 = - 0.0075 m
h_1 = - 0.0025 m
h_2 = 0.0025 m
h_3 = 0.0075 m



4.2. Stress-strain Relations for a Laminate

Step 4: Find three stiffness matrices [A], [B], and [D]

$$A_{ij} = \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \; (h_k - h_{k-1}),\; i = 1,2,6;\; j = 1,2,6$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \; (h_k^2 - h_{k-1}^2),\; i = 1,2,6;\; j = 1,2,6$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \; (h_k^3 - h_{k-1}^3),\; i = 1, 2, 6;\; j = 1, 2, 6$$

4.2. Stress-strain Relations for a Laminate

The extensional stiffness matrix [A] is $A_{ij} = \sum_{k=1}^{3} [\overline{Q}_{ij}]_k \, (h_k - h_{k-1})$

$$[A] = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)\,[(-0.0025) - (-0.0075)]$$

$$+ \begin{bmatrix} 109.4 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix} (10^9)\,[0.0025 - (-0.0025)]$$

$$+ \begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix} (10^9)\,[0.0075 - 0.0025]$$

$$[A] = \begin{bmatrix} 1.739(10^9) & 3.884(10^8) & 5.663(10^7) \\ 3.884(10^8) & 4.533(10^8) & -1.141(10^8) \\ 5.663(10^7) & -1.141(10^8) & 4.525(10^8) \end{bmatrix} \text{Pa.m}$$

The coupling stiffness matrix [B] is $B_{ij} = \frac{1}{2}\sum_{k=1}^{3}[\overline{Q}_{ij}]_k\,(h_k^2 - h_{k-1}^2)$

$$[B] = \frac{1}{2}\begin{bmatrix} 181.2 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\,[(-0.0025)^2 - (-0.0075)^2\,)]$$

$$+\frac{1}{2}\begin{bmatrix} 109.4 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix}(10^9)\left[(0.0025)^2 - (-0.0025)^2\right]$$

$$+\frac{1}{2}\begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix}(10^9)\,[(0.0075)^2 - (0.0025)^2]$$

$$[B] = \begin{bmatrix} -3.129\left(10^6\right) & 9.855\left(10^5\right) & -1.072\left(10^6\right) \\ 9.855\left(10^5\right) & 1.158\left(10^6\right) & -1.072\left(10^6\right) \\ -1.072\left(10^6\right) & -1.072\left(10^6\right) & 9.855\left(10^5\right) \end{bmatrix} Pa.m^2$$

The bending stiffness matrix [D] is $D_{ij} = \frac{1}{3}\sum_{k=1}^{3}[\overline{Q}_{ij}]_k (h_k^3 - h_{k-1}^3)$

$$[D] = \frac{1}{3}\begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\left[(-0.0025)^3 - (-0.0075)^3\right]$$

$$+\ \frac{1}{3}\begin{bmatrix} 109.4 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix}(10^9)\left[(0.0025)^3 - (-0.0025)^3\right]$$

$$+\ \frac{1}{3}\begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix}(10^9)\left[(0.0075)^3 - (0.0025)^3\right]$$

$$[D] = \begin{bmatrix} 3.343(10^4) & 6.461(10^3) & -5.240(10^3) \\ 6.461(10^3) & 9.320(10^3) & -5.596(10^3) \\ -5.240(10^3) & -5.596(10^3) & 7.663(10^3) \end{bmatrix} \text{Pa.m}^3$$

b) Since the applied load is $N_x = N_y = 1000$ N/m, the mid-plane strains and curvatures can be found by solving the following set of simultaneous linear equations

$$
\begin{bmatrix} 1000 \\ 1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
=
\begin{bmatrix}
1.739(10^9) & 3.884(10^8) & 5.663(10^7) & -3.129(10^6) & 9.855(10^5) & -1.072(10^6) \\
3.884(10^8) & 4.533(10^8) & -1.141(10^8) & 9.855(10^5) & 1.158(10^6) & -1.072(10^6) \\
5.663(10^7) & -1.141(10^8) & 4.525(10^8) & -1.072(10^6) & -1.072(10^6) & 9.855(10^5) \\
-3.129(10^6) & 9.855(10^5) & -1.072(10^6) & 3.343(10^4) & 6.461(10^3) & -5.240(10^3) \\
9.855(10^5) & 1.158(10^6) & -1.072(10^6) & 6.461(10^3) & 9.320(10^3) & -5.596(10^3) \\
-1.072(10^6) & -1.072(10^6) & 9.855(10^5) & -5.240(10^3) & -5.596(10^3) & 7.663(10^3)
\end{bmatrix}
\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}
$$

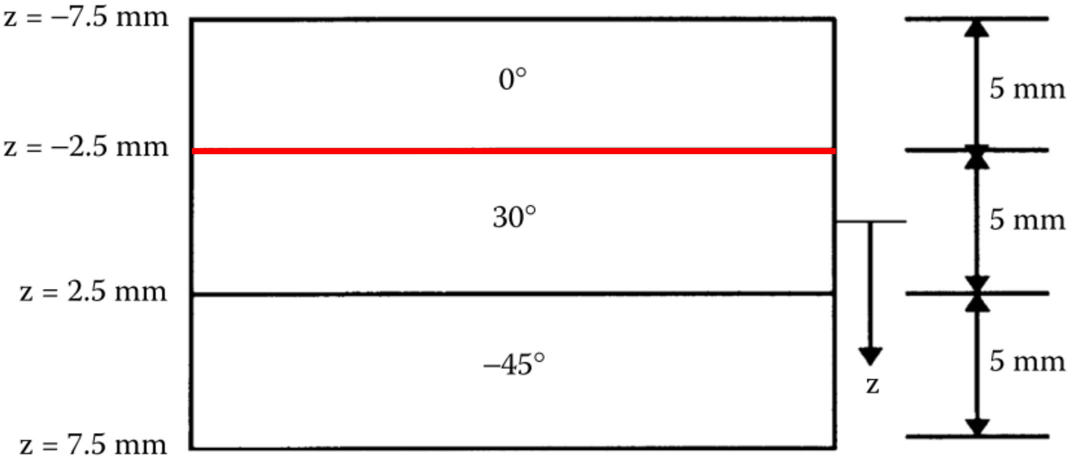
4.2. Stress-strain Relations for a Laminate

This gives

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{matrix} \begin{bmatrix} 3.123(10^{-7}) \\ 3.492(10^{-6}) \\ -7.598(10^{-7}) \end{bmatrix} \mathrm{m/m} \\ \hline \begin{bmatrix} 2.971(10^{-5}) \\ -3.285(10^{-4}) \\ 4.101(10^{-4}) \end{bmatrix} 1/\mathrm{m} \end{matrix}$$

c) The strains and stresses at the top surface of the 30^0 ply are found as follows. The top surface of the 30^0 ply is located at $z = h_1 = -0.0025$ m. Global strains at the top surface of the 30^0 ply are

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_{30^0, top} = \begin{bmatrix} 3.123(10^{-7}) \\ 3.492(10^{-6}) \\ -7.598(10^{-7}) \end{bmatrix} + (-0.0025) \begin{bmatrix} 2.971(10^{-5}) \\ -3.285(10^{-4}) \\ 4.101(10^{-4}) \end{bmatrix} =$$



$$= \begin{bmatrix} 2.380(10^{-7}) \\ 4.313(10^{-6}) \\ -1.785(10^{-6}) \end{bmatrix} \mathrm{m/m}$$

Global strains (m/m)

Ply #	Position	ε_x	ε_y	γ_{xy}
1 (0^0)	Top	$8.944\ (10^{-8})$	$5.955\ (10^{-6})$	$-3.836\ (10^{-6})$
	Middle	$1.637\ (10^{-7})$	$5.134\ (10^{-6})$	$-2.811\ (10^{-6})$
	Bottom	$2.380\ (10^{-7})$	$4.313\ (10^{-6})$	$-1.785\ (10^{-6})$
2 (30^0)	Top	$2.380\ (10^{-7})$	$4.313\ (10^{-6})$	$-1.785\ (10^{-6})$
	Middle	$3.123\ (10^{-7})$	$3.492\ (10^{-6})$	$-7.598\ (10^{-7})$
	Bottom	$3.866\ (10^{-7})$	$2.670\ (10^{-6})$	$2.655\ (10^{-7})$
3(-45^0)	Top	$3.866\ (10^{-7})$	$2.670\ (10^{-6})$	$2.655\ (10^{-7})$
	Middle	$4.609\ (10^{-7})$	$1.849\ (10^{-6})$	$1.291\ (10^{-6})$
	Bottom	$5.352\ (10^{-7})$	$1.028\ (10^{-6})$	$2.316\ (10^{-6})$

Global stresses in 30° ply

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}_{30^0,\ top}=\begin{bmatrix}109.4 & 32.46 & 54.19\\ 32.46 & 23.65 & 20.05\\ 54.19 & 20.05 & 36.74\end{bmatrix}(10^9)\begin{bmatrix}2.380(10^{-7})\\ 4.313(10^{-6})\\ -1.785(10^{-6})\end{bmatrix}$$

$$=\begin{bmatrix}6.930(10^4)\\ 7.391(10^4)\\ 3.381(10^4)\end{bmatrix}\text{Pa}$$

Note

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}=\begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}$$

4.2. Stress-strain Relations for a Laminate

Global stresses in Pa

Ply #	Position	σ_x	σ_y	τ_{xy}
1 (0^0)	Top	$3.351\ (10^4)$	$6.188\ (10^4)$	$-2.750\ (10^4)$
	Middle	$4.464\ (10^4)$	$5.359\ (10^4)$	$-2.015\ (10^4)$
	Bottom	$5.577\ (10^4)$	$4.531\ (10^4)$	$-1.280\ (10^4)$
2 (30^0)	Top	$6.930\ (10^4)$	$7.391\ (10^4)$	$3.381\ (10^4)$
	Middle	$1.063\ (10^5)$	$7.747\ (10^4)$	$5.903\ (10^4)$
	Bottom	$1.434\ (10^5)$	$8.102\ (10^4)$	$8.426\ (10^4)$
3 (-45^0)	Top	$1.235\ (10^5)$	$1.563\ (10^5)$	$-1.187\ (10^5)$
	Middle	$4.903\ (10^4)$	$6.894\ (10^4)$	$-3.888\ (10^4)$
	Bottom	$-2.547\ (10^4)$	$-1.840\ (10^4)$	$4.091\ (10^4)$

Local Strains at top of 30º ply

The local strains in the 30⁰ ply at the top surface are found using transformation equations as

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0.7500 & 0.2500 & 0.8660 \\ 0.2500 & 0.7500 & -0.8660 \\ -0.4330 & 0.4330 & 0.5000 \end{bmatrix}}^{[T]} \begin{bmatrix} 2.380(10^{-7}) \\ 4.313(10^{-6}) \\ -1.785(10^{-6})/2 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = [R][T][R]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 4.837(10^{-7}) \\ 4.067(10^{-6}) \\ 2.636(10^{-6}) \end{bmatrix} \mathrm{m/m}$$

Note

$$T = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2-s^2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4.2. Stress-strain Relations for a Laminate

Local Strains (m/m)

Ply #	Position	ε_1	ε_2	γ_{12}
1 (0^0)	Top	$8.944\,(10^{-8})$	$5.955(10^{-6})$	$-3.836(10^{-6})$
	Middle	$1.637\,(10^{-7})$	$5.134(10^{-6})$	$-2.811(10^{-6})$
	Bottom	$2.380\,(10^{-7})$	$4.313(10^{-6})$	$-1.785(10^{-6})$
2 (30^0)	Top	$4.837(10^{-7})$	$4.067(10^{-6})$	$2.636(10^{-6})$
	Middle	$7.781(10^{-7})$	$3.026(10^{-6})$	$2.374(10^{-6})$
	Bottom	$1.073(10^{-6})$	$1.985(10^{-6})$	$2.111(10^{-6})$
3 (-45^0)	Top	$1.396(10^{-6})$	$1.661(10^{-6})$	$-2.284(10^{-6})$
	Middle	$5.096(10^{-7})$	$1.800(10^{-6})$	$-1.388(10^{-6})$
	Bottom	$-3.766(10^{-7})$	$1.940(10^{-6})$	$-4.928(10^{-7})$

Local Stresses at top of 30º ply

The local stress in the 30⁰ ply at the top surface are found using transformation equations as

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix} = \overbrace{\begin{bmatrix}0.7500 & 0.2500 & .8660\\ 0.2500 & 0.7500 & -.8660\\ -0.4330 & 0.4330 & 0.5000\end{bmatrix}}^{[T]} \begin{bmatrix}6.930(10^4)\\ 7.391(10^4)\\ 3.381(10^4)\end{bmatrix}$$

$$= \begin{bmatrix}9.973(10^4)\\ 4.348(10^4)\\ 1.890(10^4)\end{bmatrix} \text{Pa}$$

Note

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix} = \overbrace{\begin{bmatrix}c^2 & s^2 & 2sc\\ s^2 & c^2 & -2sc\\ -sc & sc & c^2-s^2\end{bmatrix}}^{[T]} \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

Local Stresses in Pa

Ply #	Position	σ_1	σ_2	τ_{12}
1 (0^0)	Top	$3.351\ (10^4)$	$6.188\ (10^4)$	$-2.750\ (10^4)$
	Middle	$4.464\ (10^4)$	$5.359(10^4)$	$-2.015\ (10^4)$
	Bottom	$5.577\ (10^4)$	$4.531\ (10^4)$	$-1.280\ (10^4)$
2 (30^0)	Top	$9.973\ (10^4)$	$4.348\ (10^4)$	$1.890\ (10^4)$
	Middle	$1.502\ (10^5)$	$3.356\ (10^4)$	$1.702\ (10^4)$
	Bottom	$2.007\ (10^5)$	$2.364\ (10^4)$	$1.513\ (10^4)$
3 (-45^0)	Top	$2.586\ (10^5)$	$2.123\ (10^4)$	$-1.638\ (10^4)$
	Middle	$9.786\ (10^4)$	$2.010\ (10^4)$	$-9.954\ (10^3)$
	Bottom	$-6.285\ (10^4)$	$1.898\ (10^4)$	$-3.533\ (10^3)$

4.2. Stress-strain Relations for a Laminate

d) Portion of load taken by each ply

σ_x at the middle surface

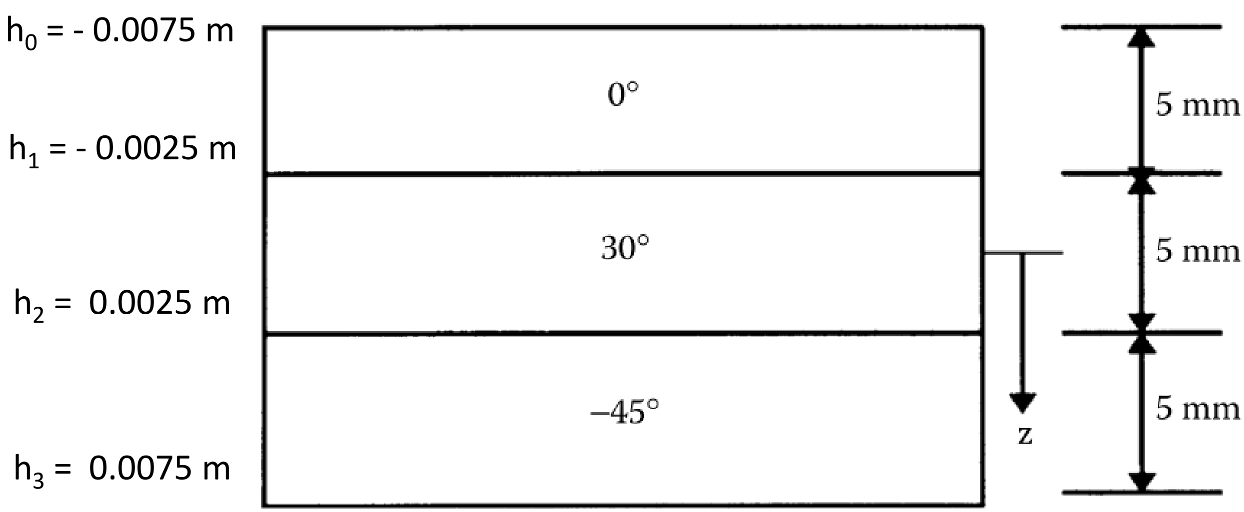
Thickness of the ply

Portion of load N_x taken by 0^0 ply = $(4.464 \times 10^4)(5 \times 10^{-3})$ = 223.2 N/m

Portion of load N_x taken by 30^0 ply = $(1.063 \times 10^5)(5 \times 10^{-3})$ = 531.5 N/m

Portion of load N_x taken by -45^0 ply = $(4.903 \times 10^4)(5 \times 10^{-3})$= 245.2 N/m

The sum total of the loads shared by each ply is 1000 N/m, (223.2 + 531.5 + 245.2) which is the applied load in the x-direction, N_x.



$$N_x^0 = \int_{h_1}^{h_0} \sigma_x dz$$

$$N_x^{30} = \int_{h_2}^{h_1} \sigma_x dz$$

$$N_x^{-45} = \int_{h_3}^{h_2} \sigma_x dz$$

Percentage of load N_x taken by 0^0 ply $= \dfrac{223.2}{1000} \times 100$

$= 22.32\ \%$

Percentage of load N_x taken by 30^0 ply $= \dfrac{531.5}{1000} \times 100$

$= 53.15\ \%$

Percentage of load N_x taken by -45^0 ply $= \dfrac{245.2}{1000} \times 100$

$= 24.52\ \%$

Compliance Matrices of a Laminate

1. Since multidirectional laminates are characterized by stress discontinuities from ply to ply, therefore **it is preferable to work with strains** which are continuous through the thickness.
2. For this reason it is necessary to invert the load-deformation relations and express strains and curvatures as a function of applied loads and moments.

$$\begin{bmatrix} \underline{N} \\ M \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} \\ B & D \end{bmatrix} \begin{bmatrix} \underline{\varepsilon^0} \\ \kappa \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{\varepsilon^0} \\ \kappa \end{bmatrix} = \begin{bmatrix} \underline{A^*} & \underline{B^*} \\ C^* & D^* \end{bmatrix} \begin{bmatrix} \underline{N} \\ M \end{bmatrix}$$

$$\begin{bmatrix} \underline{A^*} & \underline{B^*} \\ C^* & D^* \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} \\ B & D \end{bmatrix}^{-1} \qquad \left[C^*\right] = \left[B^*\right]^T$$

The [A*], [B*], and [D*] matrices are called the extensional compliance matrix, coupling compliance matrix, and bending compliance matrix respectively.

In-Plane Engineering Constants of a Laminate

For a symmetric laminate:

$$[B] = 0, \qquad A_{ij} = \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \, (h_k - h_{k-1}),\ i = 1, 2, 6;\ j = 1, 2, 6$$

$$[A^*] = [A]^{-1}, \qquad B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \, (h_k^2 - h_{k-1}^2),\ i = 1, 2, 6;\ j = 1, 2, 6$$

$$[D^*] = [D]^{-1} \qquad D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \, (h_k^3 - h_{k-1}^3)\ i = 1,2,6;\ j = 1,2,6$$

$$\left[\frac{\varepsilon^0}{\kappa}\right] = \left[\begin{array}{c|c} A^* & B^* \\ \hline C^* & D^* \end{array}\right] \left[\frac{N}{M}\right] \Rightarrow \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}$$

In-Plane Engineering Constants of a Laminate

Effective in-plane longitudinal modulus E_x

$$N_x \neq 0, N_y = 0, N_{xy} = 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} N_x \\ 0 \\ 0 \end{bmatrix}$$

$$\varepsilon_x^0 = A_{11}^* N_x$$

$$\boxed{E_x \equiv \frac{\sigma_x}{\varepsilon_x^0} = \frac{N_x / h}{A_{11}^* N_x} = \frac{1}{h A_{11}^*}}$$

Effective in-plane transverse modulus E_y

$$N_x = 0, N_y \neq 0, N_{xy} = 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} 0 \\ N_y \\ 0 \end{bmatrix}$$

$$\varepsilon_y^0 = A_{22}^* N_y$$

$$\boxed{E_y \equiv \frac{\sigma_y}{\varepsilon_y^0} = \frac{N_y / h}{A_{22}^* N_y} = \frac{1}{h A_{22}^*}}$$

In-Plane Engineering Constants of a Laminate

Effective in-plane shear modulus G_{xy}

$$N_x = 0, N_y = 0, N_{xy} \neq 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ N_{xy} \end{bmatrix}$$

$$\gamma_{xy}^0 = A_{66}^* N_{xy}$$

$$\boxed{G_{xy} \equiv \frac{\tau_{xy}}{\gamma_{xy}^0} = \frac{N_{xy}/h}{A_{66}^* N_{xy}} = \frac{1}{h A_{66}^*}}$$

Effective in-plane Poisson's ration ν_{xy}

$$N_x \neq 0, N_y = 0, N_{xy} = 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} N_x \\ 0 \\ 0 \end{bmatrix}$$

$$\varepsilon_y^0 = A_{12}^* N_x$$

$$\varepsilon_x^0 = A_{11}^* N_x$$

$$\boxed{\nu_{xy} \equiv -\frac{\varepsilon_y^0}{\varepsilon_x^0} = -\frac{A_{12}^* N_x}{A_{11}^* N_x} = -\frac{A_{12}^*}{A_{11}^*}}$$

In-Plane Engineering Constants of a Laminate

Effective in-plane Poisson's ration ν_{yx}

$$N_x = 0, N_y \neq 0, N_{xy} = 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} 0 \\ N_y \\ 0 \end{bmatrix}$$

$$\varepsilon_x^0 = A_{12}^* N_y \qquad \varepsilon_y^0 = A_{22}^* N_y$$

$$\boxed{\nu_{yx} \equiv -\frac{\varepsilon_x^0}{\varepsilon_y^0} = -\frac{A_{12}^* N_y}{A_{22}^* N_y} = -\frac{A_{12}^*}{A_{22}^*}}$$

$$\frac{\nu_{xy}}{E_x} = \left(-\frac{A_{12}^*}{A_{11}^*}\right) h A_{11}^*$$

$$= -A_{12}^* h$$

$$\frac{\nu_{yx}}{E_y} = \left(-\frac{A_{12}^*}{A_{22}^*}\right) h A_{22}^*$$

$$= -A_{12}^* h$$

$$\boxed{\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}}$$

Flexural Engineering Constants of a Laminate

For a symmetric laminate: $[\mathrm{B}] = 0$

$$\left[\frac{\varepsilon^{0}}{\kappa}\right] = \left[\begin{array}{cc} A^{*} & B^{*} \\ \hline C^{*} & D^{*} \end{array}\right] \left[\frac{N}{M}\right]$$

$$[C^{*}] = [B^{*}]^{T}$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} D^{*}_{11} & D^{*}_{12} & D^{*}_{16} \\ D^{*}_{12} & D^{*}_{22} & D^{*}_{26} \\ D^{*}_{16} & D^{*}_{26} & D^{*}_{66} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}$$

Flexural Engineering Constants of a Laminate

Effective flexural longitudinal modulus E_x^f

$$M_x \neq 0, M_y = 0, M_{xy} = 0$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} \begin{bmatrix} M_x \\ 0 \\ 0 \end{bmatrix}$$

$$\kappa_x = D_{11}^* M_x$$

$$E_x^f \equiv \frac{12 M_x}{\kappa_x h^3} \Rightarrow \boxed{E_x^f = \frac{12}{h^3 D_{11}^*}}$$

Flexural Engineering Constants of a Laminate

Similarly, other flexural elastic constants can be obtained

$$E_y^f = \frac{12}{h^3 D_{22}^*}$$

$$G_{xy}^f = \frac{12}{h^3 D_{66}^*}$$

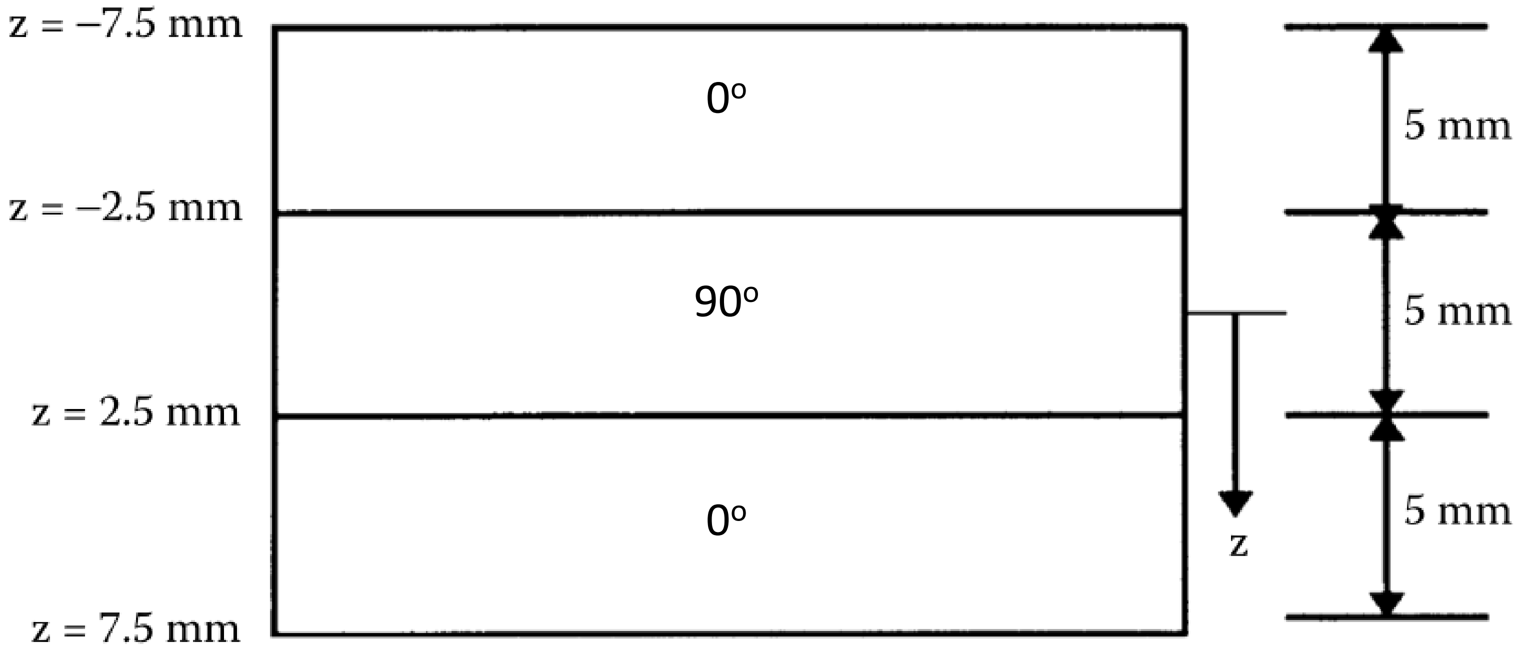
$$\nu_{xy}^f = - \frac{D_{12}^*}{D_{11}^*}$$

$$\nu_{yx}^f = - \frac{D_{12}^*}{D_{22}^*}$$

$$\frac{\nu_{xy}^f}{E_x^f} = \frac{\nu_{yx}^f}{E_y^f}$$

Example 2

Find the in-plane and flexural stiffness constants for a three-ply graphite/epoxy laminate. Use the unidirectional properties of graphite/epoxy from Table 2.1 (Autar K. Kaw, Mechanics of Composite Materials, 2nd edition, Taylor & Francis, 2006). Each lamina is 5 mm thick.



Solution

From Example 1, the transformed reduced stiffness matrix is

$$[\bar{Q}]_0 = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)\ Pa$$

Therefore, the transformed reduced stiffness matrix is

$$[\bar{Q}]_{90} = \begin{bmatrix} 10.35 & 2.897 & 0 \\ 2.897 & 181.8 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)\ Pa$$

The total thickness of the laminate is h = 0.005 × 3 = 0.015 m. The mid-plane is 0.0075 m from the top and bottom surfaces of the laminate. Thus,

$$\boxed{\begin{aligned} h_0 &= -0.0075\ m \\ h_1 &= -0.0025\ m \\ h_2 &= 0.0025\ m \\ h_3 &= 0.0075\ m \end{aligned}}$$

4.3. Engineering Constants for a Laminate

The extensional stiffness matrix [A] is $A_{ij} = \sum_{k=1}^{3} [\overline{Q}_{ij}]_k (h_k - h_{k-1})$

$$[A] = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)[-0.0025 - (-0.0075)]$$

$$+ \begin{bmatrix} 10.35 & 2.897 & 0 \\ 2.897 & 181.8 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)[0.0025 - (-0.0025)]$$

$$+ \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)[0.0075 - 0.0025]$$

4.3. Engineering Constants for a Laminate

The extensional stiffness matrix [A] is

$$[A]=\begin{bmatrix} 1.870\times10^{9} & 4.345\times10^{7} & 0 \\ 4.345\times10^{7} & 1.013\times10^{9} & 0 \\ 0 & 0 & 1.076\times10^{8} \end{bmatrix} Pa-m$$

Inverting the extensional stiffness matrix [A], we get the extensional compliance matrix [A*]

$$[A^{*}]=\begin{bmatrix} 5.353\times10^{-10} & -2.297\times10^{-11} & 0 \\ -2.297\times10^{-11} & 9.886\times10^{-10} & 0 \\ 0 & 0 & 9.298\times10^{-9} \end{bmatrix} \frac{1}{Pa-m}$$

The in-plane engineering constants (elastic moduli)

$$E_x = \frac{1}{hA^*_{11}} = \frac{1}{(0.015)(5.353\times 10^{-10})} = 124.5\,GPa$$

$$E_y = \frac{1}{hA^*_{22}} = \frac{1}{(0.015)(9.886\times 10^{-10})} = 67.43\,GPa$$

$$G_{xy} = \frac{1}{hA^*_{66}} = \frac{1}{(0.015)(9.289\times 10^{-9})} = 7.17\,GPa$$

$$\nu_{xy} = -\frac{A^*_{12}}{A^*_{11}} = -\frac{-2.297\times 10^{-11}}{5.353\times 10^{-10}} = 0.04292$$

$$\nu_{yx} = -\frac{A^*_{12}}{A^*_{22}} = -\frac{-2.297\times 10^{-11}}{9.886\times 10^{-10}} = 0.02323$$

4.3. Engineering Constants for a Laminate

The bending stiffness matrix [D] is $D_{ij} = \dfrac{1}{3}\sum_{k=1}^{3}[\overline{Q}_{ij}]_k(h_k^3 - h_{k-1}^3)$

$$[D] = \frac{1}{3}\begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)[(-0.0025)^3 - (-0.0075)^3]$$

$$+\frac{1}{3}\begin{bmatrix} 10.35 & 2.897 & 0 \\ 2.897 & 181.8 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)[(0.0025)^3 - (-0.0025)^3]$$

$$+\frac{1}{3}\begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)[(0.0075)^3 - (0.0025)^3]$$

4.3. Engineering Constants for a Laminate

The bending stiffness matrix [D] is

$$[D]=\begin{bmatrix} 4.935\times10^{4} & 8.148\times10^{2} & 0 \\ 8.148\times10^{2} & 4.696\times10^{3} & 0 \\ 0 & 0 & 2.017\times10^{3} \end{bmatrix} Pa-m^{3}$$

Inverting the bending stiffness matrix [D], we get the bending compliance matrix [D*] is

$$[D^{*}]=\begin{bmatrix} 2.032\times10^{-5} & -3.526\times10^{-6} & 0 \\ -3.526\times10^{-6} & 2.136\times10^{-4} & 0 \\ 0 & 0 & 4.959\times10^{-4} \end{bmatrix} \frac{1}{Pa-m^{3}}$$

The flexural engineering constants (flexural elastic moduli)

$$E_x^f = \frac{12}{h^3 D_{11}^*} = \frac{12}{(0.015)^3 (2.032 \times 10^{-5})} = 175.0\, GPa$$

$$E_y^f = \frac{12}{h^3 D_{22}^*} = \frac{12}{(0.015)^3 (2.136 \times 10^{-4})} = 16.65\, GPa$$

$$G_{xy}^f = \frac{12}{h^3 D_{66}^*} = \frac{12}{(0.015)^3 (4.959 \times 10^{-4})} = 7.17\, GPa$$

$$\nu_{xy}^f = -\frac{D_{12}^*}{D_{11}^*} = -\frac{-3.526 \times 10^{-6}}{2.032 \times 10^{-5}} = 0.1735$$

$$\nu_{yx}^f = -\frac{D_{12}^*}{D_{22}^*} = -\frac{-3.526 \times 10^{-6}}{2.136 \times 10^{-4}} = 0.01651$$

4.4. Hygrothermal Effects in a Laminate

- Sources of hygrothermal loads include cooling down from processing temperatures, operating temperatures different from processing temperatures, and humid environments such as in aircraft flying at high altitudes. The mechanical strains are

$$\begin{bmatrix} \varepsilon_x^M \\ \varepsilon_y^M \\ \gamma_{xy}^M \end{bmatrix}_k = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k - \begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix}_k - \begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix}_k =$$

$$= \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k - \Delta T \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k - \Delta C \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}_k$$

where the superscript *M* represents the mechanical strains, *T* stands for the free expansion thermal strain, and *C* refers to the free expansion moisture strains.

4.4.1. Hygrothermal Stresses and Strains

- Assuming that the mechanical strains induced by hygrothermal loads alone.
- Using stress–strain equation for 2D lamina (Chapter 2), the hygrothermal stresses in a lamina are then given by

$$\begin{bmatrix} \sigma_x^{TC} \\ \sigma_y^{TC} \\ \tau_{xy}^{TC} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^{M} \\ \varepsilon_y^{M} \\ \gamma_{xy}^{M} \end{bmatrix}_k$$

where TC stands for combined thermal and moisture effects.

Hygrothermal stresses induce zero resultant forces and moments in the laminate and thus in the n-ply laminate we have

$$\int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x^{TC} \\ \sigma_y^{TC} \\ \tau_{xy}^{TC} \end{bmatrix} dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x^{TC} \\ \sigma_y^{TC} \\ \tau_{xy}^{TC} \end{bmatrix}_k dz = 0$$

$$\int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x^{TC} \\ \sigma_y^{TC} \\ \tau_{xy}^{TC} \end{bmatrix} zdz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x^{TC} \\ \sigma_y^{TC} \\ \tau_{xy}^{TC} \end{bmatrix}_k zdz = 0$$

From three equations above, we have

$$\sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}_k \begin{bmatrix}\varepsilon_x^M\\ \varepsilon_y^M\\ \gamma_{xy}^M\end{bmatrix}_k dz = 0$$

and

$$\sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}_k \begin{bmatrix}\varepsilon_x^M\\ \varepsilon_y^M\\ \gamma_{xy}^M\end{bmatrix}_k z\,dz = 0$$

Substituting the stress-strain equation and laminate strains into Eq. above.

$$\sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} - \begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix}_k - \begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix}_k \right) dz = 0$$

$$\sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} - \begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix}_k - \begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix}_k \right) z\,dz = 0$$

4.4. Hygrothermal Effects in a Laminate

$$\Rightarrow \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) dz =$$

$$= \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} \right) dz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix} \right) dz$$

$$\Rightarrow \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) zdz =$$

$$= \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} \right) zdz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix} \right) zdz$$

$$\Rightarrow \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) dz =$$

$$\boxed{\begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} = \Delta T \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}}$$

$$= \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\Delta T \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k \right) dz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\Delta C \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}_k \right) dz$$

$$\Rightarrow \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) z\,dz =$$

$$\boxed{\begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix} = \Delta C \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}}$$

$$= \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\Delta T \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k \right) z\,dz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(\Delta C \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}_k \right) z\,dz$$

4.4. Hygrothermal Effects in a Laminate

Equations – from zero resultant forces and moments in the laminate

$$\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} + \begin{bmatrix} N_x^C \\ N_y^C \\ N_{xy}^C \end{bmatrix}$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix} + \begin{bmatrix} M_x^C \\ M_y^C \\ M_{xy}^C \end{bmatrix}$$

4.4. Hygrothermal Effects in a Laminate

The four arrays on the right-hand sides of 2 Eqs above are given by

$$[N^T] = \begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} = \Delta T \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k (h_k - h_{k-1})$$

$$[M^T] = \begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix} = \frac{1}{2} \Delta T \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k (h_k^2 - h_{k-1}^2)$$

$$[N^C] = \begin{bmatrix} N_x^C \\ N_y^C \\ N_{xy}^C \end{bmatrix} = \Delta C \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}_k (h_k - h_{k-1})$$

$$[M^C] = \begin{bmatrix} M_x^C \\ M_y^C \\ M_{xy}^C \end{bmatrix} = \frac{1}{2} \Delta C \sum_{k=1}^{n} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}_k (h_k^2 - h_{k-1}^2)$$

The loads in these equations are called fictitious hygrothermal loads and are known. One can calculate the midplane strains and curvatures by combining two equations above.

Final formula

$$\left[\frac{N^T}{M^T}\right]+\left[\frac{N^C}{M^C}\right]=\left[\frac{A\,|\,B}{B\,|\,D}\right]\left[\frac{\varepsilon^0}{\kappa}\right]$$

The global strains (actual strains) in any ply of the laminate can be calculated by

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix}+z\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix}.$$

However, it is the difference between the actual strains and the free expansion strains (mechanical strains), which results in mechanical stresses.

4.4. Hygrothermal Effects in a Laminate

The mechanical strains in the kth ply are given by

$$\begin{bmatrix} \varepsilon_x^M \\ \varepsilon_y^M \\ \gamma_{xy}^M \end{bmatrix}_k = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k - \begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix}_k - \begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix}_k = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k - \Delta T \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k - \Delta C \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}_k$$

The mechanical stresses in the kth ply are then calculated by

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^M \\ \varepsilon_y^M \\ \gamma_{xy}^M \end{bmatrix}_k$$

4.4.2. Coefficients of Thermal and Moisture Expansion of Laminates

The concept of finding coefficients of thermal and moisture expansion of laminates is again well suited only for symmetric laminates because, in this case, the coupling stiffness matrix [*B*] = 0 and no bending occurs under hygrothermal loads.

The coefficients of thermal expansion are defined as the change in length per unit length per unit change of temperature. Three coefficients of thermal expansion, one in direction x (α_x) and the others in direction y (α_y) and in the plane xy (α_{xy}), are defined for a laminate.

Assuming $\Delta T = 1$ and $\Delta C = 0$,

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix} \equiv \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix}_{\substack{\Delta C=0 \\ \Delta T=1}} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix}.$$

where [N^T] is the resultant thermal force corresponding to $\Delta T = 1$ and $\Delta C = 0$.

4.4. Hygrothermal Effects in a Laminate

The coefficients of moisture expansion (CMEs) are defined as the fractional change in strain per unit mass variation due to the moisture desorption or absorption. The CMEs are determined by measuring the moisture content change and the strain change between two moisture equilibrium states.

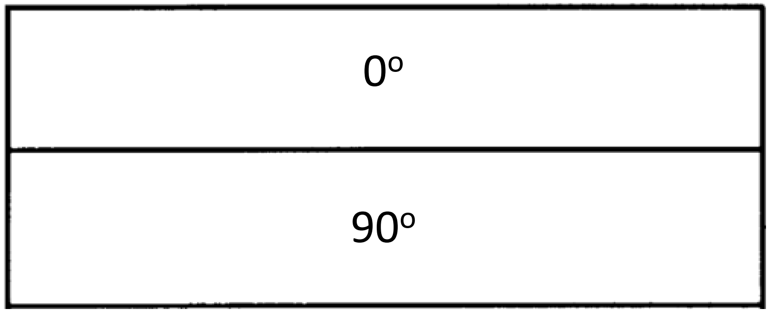
Similarly, assuming $\Delta T = 0$ and $\Delta C = 1$, the coefficients of moisture expansion can be defined as,

$$\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix} \equiv \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix}_{\substack{\Delta T=0 \\ \Delta C=1}} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} N_x^C \\ N_y^C \\ N_{xy}^C \end{bmatrix}.$$

where $[N^C]$ is the resultant moisture force corresponding to $\Delta T = 0$ and $\Delta C = 1$.

Example 3

Calculate the mechanical stresses at the bottom surface of the 90^o ply in a two-ply [0/90] Graphite/Epoxy laminate subjected to a temperature change of -75°C. Use the unidirectional properties of Graphite/Epoxy lamina from Table 2.1. Each lamina is 5 *mm* thick.



Solution

From Table 2.1, the coefficients of thermal expansion for a unidirectional graphite/epoxy ply are

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_{12} \end{bmatrix} = \begin{bmatrix} 0.200 \times 10^{-7} \\ 0.225 \times 10^{-4} \\ 0 \end{bmatrix} m/m/^oC$$

From Equation (2.181), the transformed coefficients of thermal expansion are

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{0^o} = \begin{bmatrix} 0.200 \times 10^{-7} \\ 0.225 \times 10^{-4} \\ 0 \end{bmatrix} m/m/^oC; \quad \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{90^o} = \begin{bmatrix} 0.225 \times 10^{-4} \\ 0.200 \times 10^{-7} \\ 0 \end{bmatrix} m/m/^oC$$

From Example 2, the transformed reduced stiffness matrix is

$$[\overline{Q}]_0 = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)\ Pa$$

Therefore, the transformed reduced stiffness matrix is

$$[\overline{Q}]_{90} = \begin{bmatrix} 10.35 & 2.897 & 0 \\ 2.897 & 181.8 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)\ Pa$$

The total thickness of the laminate is h = 0.005 × 2 = 0.01 m. The mid-plane is 0.000 m from the top and bottom surfaces of the laminate. Thus,

h_0 = - 0.005 m
h_1 = 0.000 m
h_2 = 0.005 m

The fictitious thermal forces are

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} = +(-75)\begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\begin{bmatrix} 0.200\ (10^{-7}) \\ 0.225\ (10^{-4}) \\ 0 \end{bmatrix}[0.000 - (-0.005)]$$

$$+(-75)\begin{bmatrix} 10.35 & 2.897 & 0 \\ 2.897 & 181.8 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\begin{bmatrix} 0.225\ (10^{4}) \\ 0.200\ (10^{7}) \\ 0 \end{bmatrix}[0.005 - 0.000]$$

$$= \begin{bmatrix} -1.131 \times 10^5 \\ -1.131 \times 10^5 \\ 0 \end{bmatrix} Pa - m.$$

$$\begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix} = \frac{1}{2}(-75)\begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\begin{bmatrix} 0.200 \times 10^{-7} \\ 0.225 \times 10^{-4} \\ 0 \end{bmatrix}[(0.000)^2 - (-0.005)^2]$$

$$+ \frac{1}{2}(-75)\begin{bmatrix} 10.35 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\begin{bmatrix} 0.225 \times 10^{-4} \\ 0.200 \times 10^{-7} \\ 0 \end{bmatrix}[(0.005)^2 - (0.000)^2]$$

$$= \begin{bmatrix} -1.538 \times 10^2 \\ 1.538 \times 10^2 \\ 0 \end{bmatrix} Pa \ m$$

Three stiffness matrices [A], [B], and [D]

$$[A] = \begin{bmatrix} 9.608\times10^{8} & 2.897\times10^{7} & 0 \\ 2.897\times10^{7} & 9.608\times10^{8} & 0 \\ 0 & 0 & 7.170\times10^{7} \end{bmatrix} Pa\text{-}m$$

$$[B] = \begin{bmatrix} -2.143\times10^{6} & 0 & 0 \\ 0 & 2.143\times10^{6} & 0 \\ 0 & 0 & 0 \end{bmatrix} Pa\text{-}m^{2}$$

$$[D] = \begin{bmatrix} 8.007\times10^{3} & 2.414\times10^{2} & 0 \\ 2.414\times10^{2} & 8.007\times10^{3} & 0 \\ 0 & 0 & 5.975\times10^{2} \end{bmatrix} Pa\text{-}m^{3}$$

The mid-plane strains and curvatures can be found as

$$\begin{bmatrix} -1.131\times10^{5} \\ -1.131\times10^{5} \\ 0 \\ -1.538\times10^{2} \\ 1.538\times10^{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 9.608\times10^{8} & 2.897\times10^{7} & 0 & -2.143\times10^{6} & 0 & 0 \\ 2.897\times10^{7} & 9.608\times10^{8} & 0 & 0 & 2.143\times10^{6} & 0 \\ 0 & 0 & 7.170\times10^{7} & 0 & 0 & 0 \\ -2.143\times10^{6} & 0 & 0 & 8.007\times10^{3} & 2.414\times10^{2} & 0 \\ 0 & 2.143\times10^{6} & 0 & 2.414\times10^{2} & 8.007\times10^{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.975\times10^{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} -3.907\times10^{-4} \\ -3.907\times10^{-4} \\ 0 \\ \hline -1.276\times10^{-1} \\ 1.276\times10^{-1} \\ 0 \end{bmatrix} \begin{matrix} m/m \\ \\ 1/m \end{matrix}$$

$$\begin{bmatrix} N^T \\ \hline M^T \end{bmatrix} = \begin{bmatrix} A & B \\ \hline B & D \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \hline \kappa \end{bmatrix}$$

The actual strains at the bottom surface (h_2 = 0.005) of the 90° ply as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_{90^o} = \begin{bmatrix} -3.907\times10^{-4} \\ -3.907\times10^{-4} \\ 0 \end{bmatrix} + (0.005)\begin{bmatrix} -1.276\times10^{-1} \\ 1.276\times10^{-1} \\ 0 \end{bmatrix} = \begin{bmatrix} -1.029\times10^{-3} \\ 2.475\times10^{-4} \\ 0 \end{bmatrix} m/m$$

The mechanical strains result in the residual stresses. Thus, if one subtracts the strains that would have been caused by free expansion from the actual strains, one can calculate the mechanical strains. The free expansion thermal strains in the 90° ply are as follows:

$$\begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix}_{90^0} = \Delta T \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{90^0} = \begin{bmatrix} 0.225\times10^{-4} \\ 0.200\times10^{-7} \\ 0 \end{bmatrix}(-75) = \begin{bmatrix} -0.16875\times10^{-2} \\ -0.15000\times10^{-5} \\ 0 \end{bmatrix} m/m$$

4.4. Hygrothermal Effects in a Laminate

The mechanical strains at the bottom of the 90° ply are thus:

$$\begin{bmatrix} \varepsilon_x^M \\ \varepsilon_y^M \\ \gamma_{xy}^M \end{bmatrix}_{90^0} = \underbrace{\begin{bmatrix} -1.029\times10^{-3} \\ 2.475\times10^{-4} \\ 0 \end{bmatrix}}_{[\varepsilon]_{90^0}} - \underbrace{\begin{bmatrix} -0.16875\times10^{-2} \\ -0.1500\times10^{-5} \\ 0 \end{bmatrix}}_{[\varepsilon^T]_{90^0}} = \begin{bmatrix} 0.6585\times10^{-3} \\ 0.2490\times10^{-3} \\ 0 \end{bmatrix}$$

The mechanical stresses at the bottom surface of the 90° ply as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_{90^0} = \underbrace{\begin{bmatrix} 10.35 & 2.897 & 0 \\ 2.897 & 181.8 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)}_{[\bar{Q}]_{90^0}} \begin{bmatrix} 0.6585\times10^{-3} \\ 0.2490\times10^{-3} \\ 0 \end{bmatrix} = \begin{bmatrix} 7.535\times10^{6} \\ 4.718\times10^{7} \\ 0 \end{bmatrix} Pa.$$

4.4. Hygrothermal Effects in a Laminate

Global Strains for Example 3

Ply no.	Position	ε_x	ε_y	γ_{xy}
1 (0°)	Top	2.475×10^{-4}	-1.029×10^{-3}	0.0
	Middle	-7.160×10^{-5}	-7.098×10^{-4}	0.0
	Bottom	-3.907×10^{-4}	-3.907×10^{-4}	0.0
2 (90°)	Top	-3.907×10^{-4}	-3.907×10^{-4}	0.0
	Middle	-7.098×10^{-4}	-7.160×10^{-5}	0.0
	Bottom	-1.029×10^{-3}	2.475×10^{-4}	0.0

Global Stresses for Example 3

Ply no.	Position	σ_y	σ_y	τ_{xy}
1 (0°)	Top	4.718×10^{7}	7.535×10^{6}	0.0
	Middle	-9.912×10^{6}	9.912×10^{6}	0.0
	Bottom	-6.701×10^{7}	1.229×10^{7}	0.0
2 (90°)	Top	1.229×10^{7}	-6.701×10^{7}	0.0
	Middle	9.912×10^{6}	-9.912×10^{6}	0.0
	Bottom	7.535×10^{6}	4.718×10^{7}	0.0

4.5. Failure Criteria for a Laminate

- Laminate failure may not be catastrophic.
- Some layer(s) fail first and the composite continues to take more loads until all the plies fail.
- Failed plies may still contribute to the stiffness and strength of the laminate.
- The degradation of the stiffness and strength properties of each failed lamina depends on the philosophy of the user:
  - When a ply fails, it may have cracks parallel to the fibers. This ply is still capable of taking load parallel to the fibers. Here, the cracked ply can be replaced by a hypothetical ply that has no transverse stiffness and strength as well as the shear strength. The longitudinal modulus and strength remain unchanged.
  - When a ply fails, fully discount the ply and replace the ply of near-zero stiffness and strength.

4.5. Failure Criteria for a Laminate

- Procedure for finding the successive loads between first ply failure and last ply failure given next follows the fully discounted method:

1. Use CLT to find the mid-plane strains and curvatures given the applied load (mechanical, thermal, moisture).
2. Find the local stresses and strains in each ply.
3. Apply the failure theories to each ply to determine the strength ratio(SR) for each ply. Multiplying the SR to the applied load gives the load level of the first ply failure.
4. Degrade fully the stiffness of damaged ply or plies. Apply the actual load level of the previous failure.
5. Repeat to find the (SR) for the undamaged plies.
   - If the SR>1, multiply to the applied load to obtain the load level of the next ply failure and repeat.
   - If the SR<1, degrade the stiffness and strength properties of all the damaged plies and apply the actual load level of the previous failure.
6. Repeat until all of the plies have failed. The load at which all of the plies have failed is called the last ply failure load.

Strength Ratio

- In a failure theory, it can be determined whether a lamina has failed if the stress state lies within the failure envelope.
- However, it does not give information about how much the load can be increased if the lamina is safe, or how much the load should be decreased if the lamina has failed.
- The strength ratio is defined as

$$SR = \frac{\text{Maximum Load Which Can Be Applied}}{\text{Load Applied}}$$

- If SR>1, then the lamina is safe and the applied stress can be increased by a factor of SR.
- If SR<1, the lamina is unsafe and the applied stress needs to be reduced by a factor of SR.

Example 4

Find the ply-by-ply failure loads for a $\left[0/9\bar{0}\right]_s$ graphite/epoxy laminate. Assume the thickness of each ply is 5 mm and use properties of unidirectional graphite/epoxy lamina from Table 2.1. The only load applied is a tensile normal load in the *x*-direction — that is, the direction parallel to the fibers in the 0° ply.

Solution

Because the laminate is symmetric and the load applied is a normal load, only the extensional stiffness matrix [A] is required. Besides, the extensional compliance matrix [A*] is

$$[A^*] = \begin{bmatrix} 5.353\times10^{-10} & -2.297\times10^{-11} & 0 \\ -2.297\times10^{-11} & 9.886\times10^{-10} & 0 \\ 0 & 0 & 9.298\times10^{-9} \end{bmatrix} \frac{1}{Pa\text{-}m},$$

4.5. Failure Criteria for a Laminate

The midplane strains for symmetric laminates subjected to N_x = 1 N/m as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \left[A^* \right] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5.353\times10^{-10} \\ -2.297\times10^{-11} \\ 0 \end{bmatrix}$$

The global strains in the top 0° ply at the top surface can be found as follows

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_{0^o,top} = \begin{bmatrix} 5.353\times10^{-10} \\ -2.297\times10^{-11} \\ 0 \end{bmatrix} + (0.0075) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5.353\times10^{-10} \\ -2.297\times10^{-11} \\ 0 \end{bmatrix}$$

4.5. Failure Criteria for a Laminate

The global stresses at the top surface of the top 0° ply as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_{0^0,\ top} = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9) \begin{bmatrix} 5.353\times10^{-10} \\ -2.297\times10^{-11} \\ 0 \end{bmatrix} = \begin{bmatrix} 97.26 \\ 1.313 \\ 0 \end{bmatrix} Pa$$

Using the transformation, the local stresses at the top surface of the top 0° ply are

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_{0^0,\ top} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 97.26 \\ 1.313 \\ 0 \end{bmatrix} = \begin{bmatrix} 97.26 \\ 1.313 \\ 0 \end{bmatrix} Pa$$

4.5. Failure Criteria for a Laminate

All the local stresses and strains in the laminate are summarized in Tables.

Local Stresses (Pa)

Ply no.	Position	σ_1	σ_2	τ_{12}
1 (0°)	Top	9.726×10^{1}	1.313×10^{0}	0.0
	Middle	9.726×10^{1}	1.313×10^{0}	0.0
	Bottom	9.726×10^{1}	1.313×10^{0}	0.0
2 (90°)	Top	-2.626×10^{0}	5.472×10^{0}	0.0
	Middle	-2.626×10^{0}	5.472×10^{0}	0.0
	Bottom	-2.626×10^{0}	5.472×10^{0}	0.0
3 (0°)	Top	9.726×10^{1}	1.313×10^{0}	0.0
	Middle	9.726×10^{1}	1.313×10^{0}	0.0
	Bottom	9.726×10^{1}	1.313×10^{0}	0.0

Local Strain

Ply no.	Position	ε_1	ε_2	τ_{12}
1 (0°)	Top	5.353×10^{-10}	-2.297×10^{-11}	0.0
	Middle	5.353×10^{-10}	-2.297×10^{-11}	0.0
	Bottom	5.353×10^{-10}	-2.297×10^{-11}	0.0
2 (90°)	Top	-2.297×10^{-11}	5.353×10^{-10}	0.0
	Middle	-2.297×10^{-11}	5.353×10^{-10}	0.0
	Bottom	-2.297×10^{-11}	5.353×10^{-10}	0.0
3 (0°)	Top	5.353×10^{-10}	-2.297×10^{-11}	0.0
	Middle	5.353×10^{-10}	-2.297×10^{-11}	0.0
	Bottom	5.353×10^{-10}	-2.297×10^{-11}	0.0

The Tsai–Wu failure theory applied to the top surface of the top 0° ply is applied as follows. The local stresses are

$$\sigma_1 = 97.26\ Pa\ , \quad \sigma_2 = 1.313\ Pa\ , \quad \tau_{12} = 0$$

4.5. Failure Criteria for a Laminate

Using the parameters H_1, H_2, H_6, H_{11}, H_{22}, H_{66}, and H_{12} in Chapter 2 (page 94), the Tsai–Wu failure theory gives the strength ratio (SR) as

$$H_1\sigma_1 + H_2\sigma_2 + H_6\tau_{12} + H_{11}\sigma_1^2 + H_{22}\sigma_2^2 + H_{66}\tau_{12}^2 + 2H_{12}\sigma_1\sigma_2 < 1$$

$$H_1 = \frac{1}{1500\times10^6} - \frac{1}{1500\times10^6} = 0\ Pa^{-1}, \qquad H_{11} = \frac{1}{(1500\times10^6)(1500\times10^6)} = 4.4444\times10^{-19}\ Pa^{-2},$$

$$H_2 = \frac{1}{40\times10^6} - \frac{1}{246\times10^6} = 2.093\times10^{-8}\ Pa^{-1},\ H_{22} = \frac{1}{(40\times10^6)(246\times10^6)} = 1.0162\times10^{-16}\ Pa^{-2},$$

$$H_6 = 0\ Pa^{-1}, \qquad H_{66} = \frac{1}{(68\times10^6)^2} = 2.1626\times10^{-16}\ Pa^{-2}.$$

Using the Mises–Hencky criterion for evaluation of H_{12}:

$$H_{12} = -0.5\left[\left(4.4444\times10^{-19}\right)\left(1.0162\times10^{-16}\right)\right]^{\frac{1}{2}} = -3.360\times10^{-18}Pa^{-2}.$$

$$(0)\ (9.726\times10^1)\ SR + (2.093\times10^{-8})\ (1.313)\ SR + (0\times0) + (4.4444\times10^{-19})\ (9.726\times10^1)^2(SR)^2 + (1.0162\times10^{-16})\ (1.313)^2(SR)^2 + (2.1626\times10^{-16})\ (0)^2 + 2(-3.360\times10^{-18})\ (9.726\times10^1)\ (1.313)(SR)^2 = 1$$

$$SR = 1.339\times10^7.$$

4.5. Failure Criteria for a Laminate

The maximum strain failure theory can also be applied to the top surface of the top 0° ply as follows. The local strains are

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 5.353\times10^{-10} \\ -2.297\times10^{-11} \\ 0 \end{bmatrix}$$

Then, according to maximum strain failure theory (Equation 2.143), the strength ratio is given by

$$SR = \min\left\{ \begin{array}{l} \left[\left(1500\times10^{6}\right)/\left(181\times10^{9}\right)\right]/\left(5.353\times10^{-10}\right), \\ \left[\left(246\times10^{6}\right)/\left(10.3\times10^{9}\right)\right]/\left(2.979\times10^{-11}\right) \end{array} \right\} = 1.548\times10^{7}$$

4.5. Failure Criteria for a Laminate

The strength ratios for all the plies in the laminate are summarized in Table using the maximum strain and Tsai–Wu failure theories.

Ply no.	Position	Maximum strain	Tsai–Wu
1 (0°)	Top	1.548×10^7 (1T)	1.339×10^7
	Middle	1.548×10^7 (1T)	1.339×10^7
	Bottom	1.548×10^7 (1T)	1.339×10^7
2 (90°)	Top	7.254×10^6 (2T)	7.277×10^6
	Middle	7.254×10^6 (2T)	7.277×10^6
	Bottom	7.254×10^6 (2T)	7.277×10^6
3 (0°)	Top	1.548×10^7 (1T)	1.339×10^7
	Middle	1.548×10^7 (1T)	1.339×10^7
	Bottom	1.548×10^7 (1T)	1.339×10^7

From above Table and using the Tsai–Wu theory, the minimum strength ratio is found for the 90° ply. This strength ratio gives the maximum value of the allowable normal load as

$$N_x = 7.277 \times 10^6 \, \frac{N}{m}$$

and the maximum value of the allowable normal stress as

$$\frac{N_x}{h} = \frac{7.277 \times 10^6}{0.015} = 0.4851 \times 10^9 \, Pa$$

The normal strain in the x-direction at this load is

$$\left(\varepsilon_x^0\right)_{\text{first ply failure}} = \left(5.353 \times 10^{-10}\right)\left(7.277 \times 10^6\right) = 3.895 \times 10^{-3}$$

4.5. Failure Criteria for a Laminate

Now, degrading the 90° ply completely involves assuming zero stiffness and strengths of the 90° lamina. Complete degradation of a ply does not allow further failure of that ply. For the undamaged plies, the laminate has two reduced stiffness matrices as

$$[Q] = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} GPa$$

and, for the damaged ply,

$$[Q] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} GPa$$

4.5. Failure Criteria for a Laminate

The extensional stiffness matrix [A] is $A_{ij} = \sum_{k=1}^{3} [\overline{Q}_{ij}]_k (h_k - h_{k-1})$

$$[A] = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} \left(10^7\right) Pa.m$$

Inverting the extensional stiffness matrix [A], we get the extensional compliance matrix [A*]

$$\left[A^*\right] = \begin{bmatrix} 5.525\times10^{-10} & -1.547\times10^{-10} & 0 \\ -1.547\times10^{-10} & 9.709\times10^{-9} & 0 \\ 0 & 0 & 1.395\times10^{-8} \end{bmatrix} \frac{1}{Pa.m}$$

which gives mid-plane strains subjected to N_x = 1 N/m as

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} 5.525\times10^{-10} & -1.547\times10^{-10} & 0 \\ -1.547\times10^{-10} & 9.709\times10^{-9} & 0 \\ 0 & 0 & 1.395\times10^{-8} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5.525\times10^{-10} \\ -1.547\times10^{-10} \\ 0 \end{bmatrix}$$

The local stresses and local strains in each layer are found using earlier techniques given in this example in the following tables.

Local stresses after first ply failure

Ply no.	Position	σ_1	σ_2	τ_{12}
1 (0°)	Top	1.0000×10^2	0.0	0.0
	Middle	1.0000×10^2	0.0	0.0
	Bottom	1.0000×10^2	0.0	0.0
2 (90°)	Top	—	—	—
	Middle	—	—	—
	Bottom	—	—	—
3 (0°)	Top	1.0000×10^2	0.0	0.0
	Middle	1.0000×10^2	0.0	0.0
	Bottom	1.0000×10^2	0.0	0.0

Local strains after first ply failure

Ply no.	Position	ε_1	ε_2	γ_{12}
1 (0°)	Top	5.25×10^{-10}	-1.547×10^{-10}	0.0
	Middle	5.525×10^{-10}	-1.547×10^{-10}	0.0
	Bottom	5.525×10^{-10}	-1.547×10^{-10}	0.0
2 (90°)	Top	—	—	—
	Middle	—	—	—
	Bottom	—	—	—
3 (0°)	Top	5.525×10^{-10}	-1.547×10^{-10}	0.0
	Middle	5.525×10^{-10}	-1.547×10^{-10}	0.0
	Bottom	5.525×10^{-10}	-1.547×10^{-10}	0.0

4.5. Failure Criteria for a Laminate

The strength ratios in each layer are also found using methods given in this example and are shown in Table.

The strength ratios after first ply failure

Ply no.	Position	Max strain	Tsai–Wu
1 (0°)	Top	1.5000×10^7 (1T)	1.5000×10^7
	Middle	1.5000×10^7(1T)	1.5000×10^7
	Bottom	1.5000×10^7(1T)	1.5000×10^7
2 (90°)	Top	—	—
	Middle	—	—
	Bottom	—	—
3 (0°)	Top	1.5000×10^7(1T)	1.5000×10^7
	Middle	1.5000×10^7(1T)	1.5000×10^7
	Bottom	1.5000×10^7(1T)	1.5000×10^7

4.5. Failure Criteria for a Laminate

From above Table and using the Tsai–Wu theory, the minimum strength ratio is found for the 0° ply. This strength ratio gives the maximum value of the allowable normal load as

$$N_x = 1.5 \times 10^7 \frac{N}{m}$$

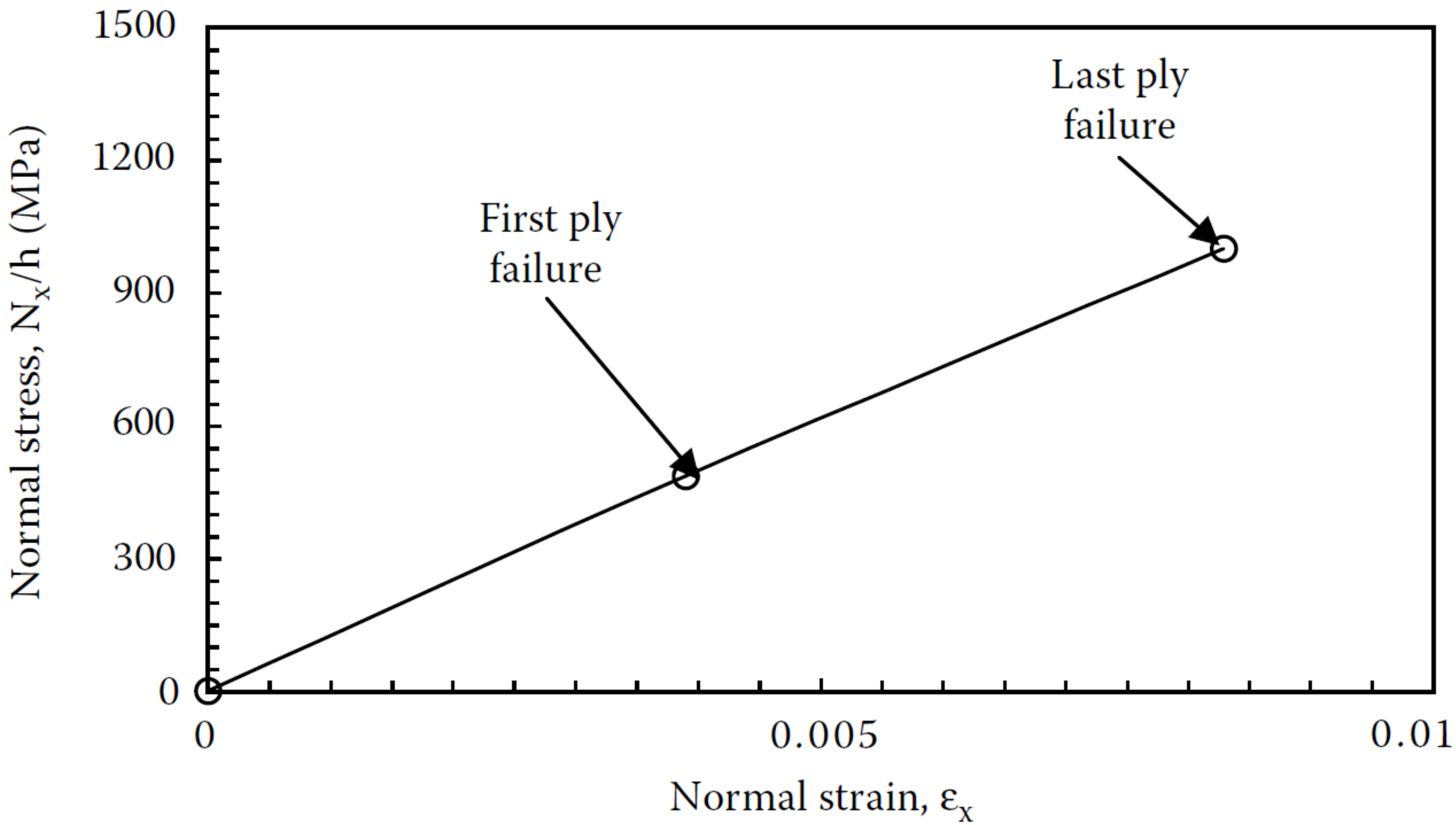
and the maximum value of the allowable normal stress as

$$\frac{N_x}{h} = \frac{1.5 \times 10^7}{0.015} = 1.0 \times 10^9 \, Pa$$

The normal strain in the x-direction at this load is

$$\left(\varepsilon_x^0\right)_{\text{last ply failure}} = \left(5.525 \times 10^{-10}\right)\left(1.5 \times 10^7\right) = 8.288 \times 10^{-3}$$

Plotting the stress vs. strain curve for the laminate until last ply failure shows that the curve will consist of two linear curves, each ending at each ply failure.



4.5. Failure Criteria for a Laminate

The slope of the two lines will be the Young's modulus in x direction for the undamaged laminate and for the first ply failure (FPF) laminate — that is, using Equation (4.35),

$$E_x = \frac{\sigma_x}{\varepsilon_x^0} = \frac{N_x / h}{A_{11}^* N_x} = \frac{1}{hA_{11}^*} = \frac{1}{(0.015)(5.353\times 10^{-10})} = 124.5\ GPa$$

until first ply failure, and

$$E_x = \frac{\sigma_x}{\varepsilon_x^0} = \frac{(N_x / h)_{\text{last ply failure}} - (N_x / h)_{\text{first ply failure}}}{(\varepsilon_x^0)_{\text{last ply failure}} - (\varepsilon_x^0)_{\text{first ply failure}}}$$

$$= \frac{0.1\times 10^{10} - 0.4851\times 10^{9}}{8.288\times 10^{-3} - 3.895\times 10^{-3}} = 117.2\ GPa$$

after first ply failure and until last ply failure (see Figure).

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4.1

Write the ply orientation code for the laminates shown below

90
0
-45
45
90
0
-45
45

0
45
0
45
90
45
0
45
0

Draw the laminate corresponding to the ply orientation codes listed below

$[(0_2/90)_2/45]_S$

$[0_2/\pm 45/\mp 45]_s$

Homework of chapter 4

4.2 For each laminate shown, determine whether it exhibits the special qualities printed below

0
90
45
90
0

Symmetric

Balanced

Quasi-isotropic

Crossply

0
0
90
90
0
0
90
90

Symmetric

Balanced

Quasi-isotropic

Crossply

0
45
90
-45
-45
90
45
0

Symmetric

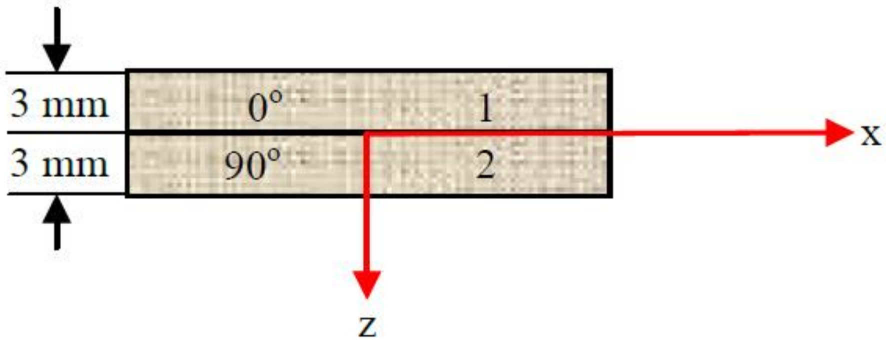
Balanced

Quasi-isotropic

Crossply

Homework of chapter 4

4.3 The 2-ply laminate is shown in the figure. Assume both the laminae have identical stiffness matrix Q as follows:



$$[Q] = \begin{bmatrix} 13 & 2.5 & 0 \\ 2.5 & 1 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \text{GPa}$$

1. Find A, B, and D matrices.
2. Calculate global stress variation in each lamina in the laminate, if the load applied is N_x = 100 kN/m.

4.4 Find the in-plane and flexural engineering elastic moduli of a $[45/–45]_s$ graphite/epoxy laminate. Each lamina is 5 mm thick. Verify the reciprocal relationships for the Poisson's ratios. Use the properties of a unidirectional graphite/epoxy lamina from Table 2.1 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006).

Homework of chapter 4

4.5 Using Tsai–Wu theory, find ply-by-ply failure load of a [45/–45]s graphite/epoxy laminate under a pure bending moment, M_x. Use properties of unidirectional graphite/epoxy lamina from Table 2.1 (A.K. Kaw. Mechanics of Composite Materials, Taylor & Francis, 2006) and assume each layer is 0.125 mm thick.

4.6 Given a [0/60/–60] glass/epoxy laminate. The engineering elastic constants of a unidirectional lamina of glass/epoxy are E_1 = 5.6 Msi, E_2 = 1.2 Msi, ν_{12} = 0.26, G_{12} = 0.6 Msi. Assume the lamina thickness to be 0.005 in. Find

1. The three stiffness matrices [A], [B], and [D].
2. The forces and moments required in the laminate to result in bending curvature of κ_x = 0.1 in.$^{-1}$ and κ_y = 0.1 in.$^{-1}$.

4.7 A $[0/60/–60]_s$ graphite/epoxy laminate subjected to a bending moment of M_x = 50 Nm/m. The engineering elastic constants of a unidirectional lamina of graphite/epoxy are E_1 = 181 GPa, E_2 = 10.3 GPa, ν_{12} = 0.28, G_{12} = 7.17 GPa. Assume the lamina thickness to be 0.5 mm. Find

1. The local stresses at the top surface of the 60° ply above the mid-plane.
2. The in-plane and flexural stiffness constants of the laminate.
3. What is the percentage of the bending moment load taken by each ply?